Groups of even type which are not of even characteristic, I

Magaard, Kay; Stroth, Gernot

DOI:
10.1007/s11856-016-1313-x

License:
Other (please specify with Rights Statement)

Document Version
Peer reviewed version

Citation for published version (Harvard):
Magaard, K & Stroth, G 2016, 'Groups of even type which are not of even characteristic, I', *Israel Journal of Mathematics*, vol. 213, no. 1, pp. 211-278. https://doi.org/10.1007/s11856-016-1313-x

GROUPS OF EVEN TYPE WHICH ARE NOT OF EVEN CHARACTERISTIC, I

KAY MAGAARD AND GERNOT STROTH

Abstract. In the ongoing revision of the classification of the finite simple groups there is a subdivision into two classes of groups, which reflects whether semisimple elements or unipotent elements are the primary focus of the investigation. While semisimple methods naturally lead to the definition of groups of even type, unipotent methods, notably the amalgam method, naturally lead to groups of even characteristic. This paper clarifies the relationship between the two definitions and thus makes the amalgam method available for use in the classification of groups of even type.

1. Introduction

One of the great achievements of the last century is the classification of the finite simple groups. The modern treatment began with the talk of R. Brauer at the International Congress of Mathematics in Amsterdam at 1954. There he suggested to classify the finite simple groups by the structure of the centralizers of their involutions. Together with the proof of the Odd Order Theorem of Feit-Thompson [FT] the strategy, which eventually was successful, for classifying the finite simple groups, was launched. In particular the prime 2 plays a prominent role. The next cornerstone in the classification of the finite simple groups is Aschbacher's Standard Component Theorem 1975 [Asch1] which shows that either $C_G(O_2(M)) \leq O_2(M)$ for all 2-local subgroups M of G or there exists an involution t such that $C_G(t)/O(C_G(t))$ possesses a subnormal $SL_2(q)$ or $C_G(t)$ is in standard form. Groups of the first type are called of characteristic 2. The groups of the second type are treated by Aschbacher's Classical Involution Theorem 1977 [Asch2]. The last case was treated by solving various standard form problems. The first case causes a lot of problems. In this case the classification tries to put the focus on some properly chosen odd prime p. Generically a group of characteristic 2 is a group of Lie type over a field of characteristic two. Then the prime p is a prime which divides the order of a torus. Using elements of order p one tries to follow arguments from before and to set

Date: February 5, 2015.

up a standard form problem now for elements of order p and continues to solve it. Unfortunately this does not work in general. So some difficult special cases like the Quasithin Group Theorem due to Aschbacher - Smith [AS] and the Uniqueness Theorem due to Aschbacher [Asch6] arise.

At present there are two strategies for revising the classification of the finite simple groups. Gorenstein-Lyons-Solomon or GLS for short generally follow the original strategy. However they have a subdivision in classes of simple groups which differs slightly from the original. They work with groups of even type rather than characteristic 2 type. We now recall the definition of even type [GoLyS1, Definition 21.3]. For this we first define a set $\mathcal{C}_2$.

Definition 1.1. [GoLyS1, Definition (12.1)(1)] The set $\mathcal{C}_2$ consists of simple and quasisimple groups.

- The simple groups in $\mathcal{C}_2$ are $K \in \mathrm{Chev}(2)$, $L_2(9)$, $L_2(p)$, p a Fermat or Mersenne prime, $L_3(3)$, $L_4(3)$, $U_4(3)$, $G_2(3)$, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HiS, Suz, Ru, Co_1, Co_2, $M(22)$, $M(23)$, $M(24)'$, Th, F_2, F_1.
- The groups $K \in \mathcal{C}_2$ with K not simple are those for which $K/O_2(K)$ is a simple group in $\mathcal{C}_2$. But the following quasisimple groups are delete, i.e. are not in $\mathcal{C}_2$: $SL_2(q)$, q odd, $2A_8$, $SL_4(3)$, $SU_4(3)$, $Sp_4(3)$, and $[X]L_3(4)$, with $X \cong \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Definition 1.2. A group G is said to be of even type if the following hold:

(i) $\mathcal{L} \subseteq \mathcal{C}_2$, where $\mathcal{L}$ is the set of all components of $C_G(x)$ for all involutions $x \in G$.
(ii) $O(C_G(x)) = 1$ for every involution $x \in G$
(iii) G has 2-rank at least 3.

In the following statement we list the finite simple groups of even type

Statement *The (known) finite simple groups of even type are the groups G in* Chev(2) *of 2-rank at least* 3*,* A_9, A_{10}, A_{12}*,* $G_2(3)$*,* $L_4(3)$*,* $PSp_4(3)$*,* $U_4(3)$*,* $\Omega_7(3)$*,* $\Omega_8^{\pm}(3)$ *and all sporadic groups with the exception of M_{11}, ON, LyS and McL.*

For the sporadic groups this is an easy inspection of [GoLyS3, (5.3)]. Recall that M_{11} is not of even type as $m_2(M_{11}) = 2 < 3$. For A_n one always has some component A_{n-4}, which gives the list. The groups $G \in \mathrm{Chev}(2)$ even satisfy that $C_G(O_2(H)) \leq O_2(H)$ for any 2-local

H of G by the Borel - Tits theorem [GoLyS3, Theorem 3.1.3]. For the groups of Lie type in odd characteristic the list will follow from the proof of Lemma 3.1 in this paper.

Meierfrankenfeld-Stellmacher-Stroth, MSS for short, follow a different strategy. They work with groups of characteristic 2 type and use 2-local subgroups rather than switching primes. Hence the main focus is to determine the structure of $M/O_2(M)$ and the action of M on $O_2(M)$ for various 2-local subgroups M to set up a parabolic system. Then using geometric methods one can eventually identify the target group G. The first main result the Structure Theorem [MeStStr] has been proved. In this approach representations of groups play an important role. Hence the basic difference, one can say roughly, is that this approach uses unipotent methods, while GLS uses more semisimple methods.

It is not so easy to verify that a group is of characteristic 2. There is a variant of this which is much easier to verify, that is even characteristic.

Definition 1.3. A group G is said to be of even characteristic, if for a Sylow 2-subgroup S and all nontrivial 2-local subgroups H of G with $S \leq H$, we have that $C_G(O_2(H)) \leq O_2(H)$.

Groups of even characteristic are sometimes also called of parabolic characteristic two. To verify that a group is of even characteristic one just has to show that for all involutions $x \in Z(S)$ one has that $C_G(O_2(C_G(x))) \leq O_2(C_G(x))$ (see Lemma 2.1). In the proof of the structure theorem the assumption that G is of characteristic 2 and not just of even characteristic has been used just at one place. At the moment there is an ongoing project to prove the theorem under the weaker assumption that G is of even characteristic.

Unfortunately we now have two projects which have incompatible definitions. Of course it would be helpful if one could easily use results from one project in the other. The aim of this paper (part I and part II) is to build this bridge. More precisely we will prove that with a few exceptions a group of even type is of even characteristic. In [AS, Chapter 16] there is a similar result under the assumption that G is quasithin. Hence we can also say that this paper is a generalization of [AS, Chapter 16].

The main theorem of both parts of this paper will be:

Theorem *Let G be a simple $\mathcal{K}_2$-group of even type. Then either G is of even characteristic or $G \cong J_1$, Co_3, $M(23)$, A_{12}, $\Omega_7(3)$ or $\Omega_8^-(3)$.*

Here we call a group G a $\mathcal{K}_2$–group if any simple factor of any nontrivial 2–local subgroup of G is either cyclic, a group of Lie type, an alternating group or one of the 26 sporadic groups. As this paper is considered to be a part of the revision of the classification of the finite simple groups, this reflects the inductive assumption of being a minimal counterexample.

For the remainder of both parts of this paper G is always a simple group of even type. Before proceeding, we say some words about quotations and how we identify the exceptional groups in the Theorem by centralizers of involutions. For the two groups of Lie type in odd characteristic we depend on the classical involution theorem [Asch2]. So from there on we may assume that there is no tightly embedded quaternion group and no subnormal $SL_2(3)$ in the centralizer of any involution. Then we use classifications of groups having a standard subgroup L. But we are not going to solve all standard subgroup problems for all groups in $\mathcal{C}_2$. This would go too far. We will just do it if $N_G(L)$ contains a Sylow 2–subgroup of G. Recall that a standard subgroup L is a component of the centralizer of some involution such that $C_G(i) \leq N_G(L)$ for all involutions $i \in C_G(L)$ and furthermore $[L, L^g] \neq 1$ for all $g \in G$. For standard subgroups L with $m_2(C_G(L)) \geq 2$, we will use [AschSe1] and [AschSe2] for the identification. But again we do not use these results in their full strength. We use them up to the point where it is proved that $N_G(L)$ could not contain a Sylow 2–subgroup of G. The case that L is a Bender group was not handled in [AschSe1] hence we include the proof in this paper (see Proposition 3.5). For the case that L is alternating we quote [Asch3]. At this point we are left with the case that $C_G(L)$ has cyclic Sylow 2–subgroups. Here we quote classifications from the literature just for the case that $Z(L)$ has even order but not for the cases $L/Z(L) \cong Sz(8)$, $L_3(4)$, $U_4(3)$ and M_{22}. For these standard components and all the remaining standard subgroups L, i.e. $Z(L) = 1$, proofs are included in this paper.

The proof proceeds as follows. Initially we assume that there is some centralizer of a 2-central involution which possesses a component. Then similar to [Asch1] we produce in Proposition 4.1 a standard subgroup L in G. Unfortunately there is no obvious reason why this standard subgroup should centralize a 2-central involution. To deal with this problem is the contents of Chapter 4. Here we trace the procedure leading to the standard subgroup very carefully and in fact show that in a counterexample to our theorem we have a standard subgroup L, maybe different from the one we constructed in the first place, which centralizes a 2-central involution. This in fact is the main theorem of this first part of the paper, which might be of independent interest. We prove:

Theorem 1.4. *Let G be a simple $\mathcal{K}_2$-group of even type. Then one of the following holds*

- *G is of even characteristic; or*
- *$G \cong \Omega_7(3)$, $\Omega_8^-(3)$ or A_{12}; or*
- *There is a 2-central involution z such that $C_G(z)$ possesses a standard subgroup L. Furthermore $C_G(L)$ is cyclic.*

In the second part of this paper we first deal with all such standard subgroups for which $|Z(L)|$ is even and the cases where L is sporadic or a group of Lie type in odd characteristic. Hence generically we have that the standard subgroup L in $C_G(z)$ is a group of Lie type in characteristic two. Now we build up a 2-local subgroup N, which is minimal with respect not to be contained in $N_G(L)$ but containing a Sylow 2-subgroup of $N_G(L)$. To determine the structure of such a group we will need the $\mathcal{K}_2$ assumption to prove results about action of N on $\Omega_1(Z(O_2(N))$. In fact this group N looks very similar to the minimal parabolic in $N_G(L)$, which is not contained in the normalizer of a root subgroup of L. The main result about this group N however is that it also contains an involution t in its center. We then prove that $C_G(t)$ also contains a standard subgroup L_1. As t and z both centralize the centralizer of a root subgroup in L, we see that both standard subgroups L and L_1 must be isomorphic. But now N is the corresponding minimal parabolic in $N_G(L_1)$, as $t \in Z(N)$. This gives N a natural meaning and so we get that the corresponding minimal parabolic N_1 in $N_G(L)$ also normalizes $Z(O_2(N))$, which shows $\langle N, N_1\rangle \leq N_G(Z(O_2(N)))$, which then leads to the final contradiction.

Acknowledgement: We thank the referee for a very careful reading and many comments which helped to improve the paper substantially.

2. Preliminaries

We start this section with some results from general group theory.

Lemma 2.1. *Let G be a group and S be a Sylow 2-subgroup of G. Then G is of even characteristic if and only if $C_G(O_2(C_G(x))) \leq O_2(C_G(x))$ for all involutions $x \in Z(S)$.*

Proof. If G is of even characteristic, then $C_G(O_2(C_G(x))) \le O_2(C_G(x))$ just by definition.

So assume now that $C_G(O_2(C_G(x))) \le O_2(C_G(x))$ for all involutions $x \in Z(S)$. Let N be some 2-local of G such that $S \le N$. We have to show that $F^*(N) = O_2(N)$. As N is a 2-local we have that $O_2(N) \ne 1$. In particular there is some involution $x \in O_2(N)$ with $[S, x] = 1$. Furthermore $[F^*(N), x] = 1$. So $F^*(N) \le C_G(x)$. Set $E = O^2(F^*(N))$. Then $[O_2(C_G(x)), E] \le O_2(C_G(x))$ and as $O_2(C_G(x)) \le S \le N$, we have $[O_2(C_G(x)), E] \le E \cap O_2(C_G(x))$. In particular we have that

$$[O(E), O_2(C_G(x))] = 1$$

and

$$[E(N), O_2(C_G(x))] \le O_2(E(N)) \le Z(E(N)).$$

By the Three-Subgroups-Lemma we get $[O_2(C_G(x)), E(N)] = 1$ and then $[O_2(C_G(x)), E] = 1$. Combining this with our hypothesis yields

$$E \le C_G(O_2(C_G(x))) \le O_2(C_G(x)).$$

So as $E = O^2(E)$, we have $E = 1$ and $F^*(N) = O_2(N)$, as asserted. □

Lemma 2.2. [Glau]*Let G be a nonabelian simple group, z an involution and $z \in S \in \mathrm{Syl}_2(G)$. Then $z^G \cap S \ne \{z\}$.*

Lemma 2.3. [GoLyS2, Lemma 15.16]*(Thompson transfer). Let G be a group, $S \in \mathrm{Syl}_2(G)$, $T \trianglelefteq S$ with $S = TA$, $A \cap T = 1$, A cyclic. If G has no subgroup of index two and u is the involution in A, then there is some $g \in G$ with $u^g \in T$ and $C_S(u^g) \in \mathrm{Syl}_2(C_G(u^g))$. In particular $|C_S(u)| \le |C_S(u^g)|$.*

Lemma 2.4. [GoLyS2, Lemma 24.1] *Let R be a p–group, p odd, and E be an elementary abelian 2-group, acting faithfully on R. Then there is a subgroup U in RE, such that U is a direct product of dihedral groups of order $2p$ and E is a Sylow 2-subgroup of U.*

Definition 2.5. A pair (X, V) is called a Goldschmidt–O'Nan pair of type (n, k) provided the following conditions hold:

(i) V is a faithful GF(2)X–module with $|V| = 2^n$.
(ii) There is a nontrivial cyclic subgroup Y of X of odd order such that if we set $V_1 = [V, Y]$, then Y acts transitively on $V_1^\sharp$.
(iii) $V_0 = C_V(Y) \ne 1$, $|V_0| = 2^k$ and if $\Omega = \{V_0^X\}$, then distinct elements of Ω intersect trivially.
(iv) $Y \trianglelefteq N_X(V_0)$.

Lemma 2.6. [GoLyS2, Proposition 14.2] *If (X, V) is a Goldschmidt–O'Nan pair of type (n, k) then one of the following holds*

(i) $\Omega = \{V_0\}$, *i.e.* V_0 *is* X*–invariant.*
(ii) $\Omega = \{V_0, V_1\}$ *and* V_0, V_1 *are interchanged by a* 2*–element in* X.
(iii) $|\Omega| = 2^{n-k}$, $\bigcup_{W \in \Omega} W^\sharp = V \setminus V_1$ *and* V_1 *is* X*–invariant.*
(iv) $k = 1$, $n = 3$ *and* X *is nonabelian of order 21.*

Lemma 2.7. *Let G be a solvable group with $O(G) = 1$. Let T be a Sylow 2-subgroup of G which is dihedral or semidihedral. If $G \neq T$ and $Z(T)$ is normal in G, then G possesses a normal subgroup $U \cong SL_2(3)$.*

Proof. First of all as $O(G) = 1$, we have $C_G(O_2(G)) \leq O_2(G)$. As $O_2(G) \neq G$, we have that there is some element ω of odd order which acts faithfully on $O_2(G)$. As T contains a cyclic subgroup of index two also $O_2(G)$ contains a cyclic subgroup of index at most two. Hence this cyclic group cannot be characteristic in $O_2(G)$. So we have that $O_2(G)$ is quaternion of order 8 or elementary abelian of order 4. The latter is not possible as $Z(T)$ is normal in G and so central in G, which would imply that ω centralizes $O_2(G)$. So we have that $O_2(G)$ is quaternion of order 8 and then $o(\omega) = 3$. Now set $U = \langle O_2(G), \omega \rangle$, the $U \cong SL_2(3)$. As $\mathrm{Out}(O_2(G)) \cong \Sigma_3$, we have that U is normal in G. $\square$

Lemma 2.8. *Let $E \cong D_8^m$ be an extraspecial group, which is a central product of m dihedral groups of order 8. Then the number of elements of order 4 in E is $2^{2m} - 2^m$.*

Proof. We prove the formula by induction. For $m = 1$ we have a dihedral group of order 8, which has exactly two elements of order 4. So let $m > 1$ and set $G = HK$, where H is dihedral of order 8, $K \cong D_8^{m-1}$ and $[H, K] = 1$. Let $u \in G$, $o(u) = 4$. Then $u = st$, with $s \in H$ and $t \in K$, $[s, t] = 1$. Set $\langle z \rangle = Z(E) = Z(H) = Z(K)$. If $o(t) = 4$, we get $s^2 = 1$. There are exactly 6 elements $s \in H$ with $s^2 = 1$. By induction there are exactly $2^{2(m-1)} - 2^{m-1}$ elements of order 4 in K. This gives $6 \times (2^{2(m-1)} - 2^{m-1})$ pairs (s, t) such that $s^2 = 1$ and $o(t) = 4$. As $szt^{-1} = st$, we get

$$3 \times (2^{2(m-1)} - 2^{m-1})$$

elements of order 4 of this kind. Let now $o(s) = 4$, then $t^2 = 1$. This then gives $2(2^{2(m-1)+1} - (2^{2(m-1)} - 2^{m-1}))$ pairs (s, t) of this kind. Again $szt = s^{-1}zt$, so we get

$$2^{2(m-1)+1} - (2^{2(m-1)} - 2^{m-1})$$

elements of order 4 in this case. Altogether we have

$$(3 \times (2^{2(m-1)} - 2^{m-1}) + 2^{2(m-1)+1} - (2^{2(m-1)} - 2^{m-1})$$

elements of order 4. But this number is $2^{2m} - 2^m$. □

Lemma 2.9. *Let E be an extraspecial group which is a central product of m dihedral groups of order 8. If there is an element ω of order 5, which induces an automorphism on E such that $C_E(\omega) = Z(E)$, then m is divisible by 4.*

Proof. We have that ω acts fixed point freely on the element of order 4 in E. Hence by Lemma 2.8 we get that

$$2^{2m} \equiv 2^m (\mod 5)$$

and so

$$2^m \equiv 1 (\mod 5).$$

So 4 divides m. □

Next we will prove some results about the groups in $\mathcal{C}_2$.

Lemma 2.10. *Let $G \cong J_2$, $M(22)$, $M(24)'$, F_2, $2F_2$ or F_1. Let v be a 2–central involution in $G \setminus Z(G)$. Then $C_G(v) \cong 2^{1+4}A_5$, $2 \cdot 2^{1+8}U_4(2) : 2$, $2^{1+12}3U_4(3) : 2$, $2^{1+22}Co_2$, $2 \cdot 2^{1+22}Co_2$ or $2^{1+24}Co_1$, respectively.*

Proof. This can be found in [GoLyS3, Table 5.3]. □

In the next lemma we will collect some properties of $L = M(23)$.

Lemma 2.11. *Set $L = M(23)$ and let T be a Sylow 2-subgroup of L. Then the following holds:*

(i) *$Z(T)$ is elementary abelian of order 4 and all involutions in L are 2-central. The centralizers of these involutions are isomorphic to:*

$$2M(22), (2 \times 2)U_6(2) : 2 \text{ or } (E_{2^2} \times D_8^4)(Z_3 \times \Omega_6^-(2)) : 2.$$

In particular if $x \in L$ is an involution then $x \in C_L(x)'$.

(ii) *$J(T)$ is elementary abelian of order 2^{11} and*

$$N_L(J(T))/J(T) \cong M_{23}.$$

Finally $N_L(J(T)$ induces orbits of length 23, 253 and 1771 on the nontrivial elements of $J(T)$.

(iii) *Let $t \in J(T)$ such that $C_L(t) \cong 2M(22)$ then*

$$N_{C_L(t)}(J(T))/J(T) \cong M_{22}$$

acts indecomposably on $J(T)$ and involves a 10-dimensional module.

(iv) *Let* $t \in J(T)$ *such that* $E(C_L(t)) \cong 2^2 \cdot U_6(2)$*, then*

$$N_{C_L(t)}(J(T))/J(T) \cong L_3(4)$$

and induces a 9*-dimensional module in* $J(T)$*.*

(v) $\mathrm{Aut}(L) = L$ *and there is no nontrivial central extension of* L*.*

Proof. (i) That $Z(T)$ is elementary abelian of order 4 and the structure of the centralizers can be found in [GoLyS3, Table 5.3u]. From this structure we easily see that $x \in C_L(x)'$ for any involution x.

(ii) That $J(T)$ is elementary abelian of order 2^{11} and the structure of $N_L(J(T))$ can be found in [Asch4, 32.3] and [GoLyS3, Table 5.3u]. The orbit length are given in [Asch4, 22.4].

(iii) The structure of $N_{C_L(t)}(J(T))$ can be found in [Asch4, 32.2]. If $J(T)$ were a direct sum of a trivial module and a 10-dimensional one, then Gaschütz Lemma would get that $C_L(t)$ splits over $\langle t \rangle$, which contradicts (i).

(iv) This can be found in [Asch4, 22.2].

(v) This follows from [GoLyS3, Table 5.3u]. □

Lemma 2.12. *Let* $G = M_{24}$ *and* V *be a faithful module for* G *with* $|V| = 2^{12}$ *and point stabilizer* M_{23}*. Then* V *is the Todd-module. Furthermore* G *induces orbits of length* 24*,* 276*,* 1771 *and* 2024 *on the nontrivial vectors in* V*. Let* $v \in V$ *such that* $|v^G| = 1771$*, then* $C_G(v)$ *is an extension of an elementary abelian group of order* 2^6 *by* $3\Sigma_6$ *and* $C_G(v)$ *induces in* $V/\langle v \rangle$ *one* 4*-dimensional, one* 6*-dimensional and one trivial module.*

Proof. The uniqueness and the orbits follow from [Asch5, chapt.19] or [Asch4, 22.1]. Here we also find that $C_G(v) \cong 2^6 3\Sigma_6$. Furthermore we find that a vector in the orbit of length 276 is centralized by $\mathrm{Aut}(M_{22})$. As neither M_{23} nor $\mathrm{Aut}(M_{22})$ contains an elementary abelian subgroup of order 64, we see that all elements in $C_V(O_2(C_G(v)))$ belong to the orbit of length 1771 and so they must be conjugate in $N_G(O_2(C_G(v)))$, which gives that $\langle v \rangle = C_V(O_2(C_G(v)))$. In particular we have that $C_{V/\langle v \rangle}(O_2(C_G(v)))$ is isomorphic to $O_2(C_G(v))$ as $C_G(v)$-module. This shows that one 6-dimensional module is involved. Now the factor module is 5-dimensional and as $C_G(v)$ cannot act trivially on this factor we also get one 4-dimensional module and one 1-dimensional module. □

Lemma 2.13. *Let* $G = M_{24}$ *and* S *be a Sylow* 2*-subgroup of* G*, then* $N_G(S) = S$*.*

Proof. By [GoLyS3, Table 5.3e] there is an involution $x \in S$ such that $C_G(x)$ is an extention of an extraspecial group of order 2^7 by $L_3(2)$.

Hence $N_G(S) \leq C_G(x)$ and, as Sylow 2-subgroups of $L_3(2)$ are self normalizing, the result follows. $\square$

Now we turn to the groups of Lie type. Most of the time we will treat groups like $Sp_4(2)'$, $G_2(2)'$ and ${}^2F_4(2)'$ together with the groups of Lie type. We therefore use the following definition.

Definition 2.14. A genuine group of Lie type in characteristic p is a group isomorphic to $O^{p'}(C_{\bar{K}}(\sigma))$, where $\bar{K}$ is a semisimple $\overline{\mathrm{GF}(p)}$-algebraic group, $\overline{\mathrm{GF}(p)}$ is the algebraic closure of $\mathrm{GF}(p)$, and σ is the Steinberg endomorphism of $\bar{K}$, see [GoLyS3, Definition 2.2.2] for details. A simple group of Lie type in characteristic p is a non-abelian composition factor of a genuine group of Lie type in characteristic p.

As $Sp_4(2)' \cong L_2(9)$, which both are elements of $\mathcal{C}_2$, we will treat this group sometimes also as a group of Lie type in odd characteristic. But this will always be clear from the context.

Lemma 2.15. (Borel-Tits-Theorem) *Let $G = G(q), q = p^f$, be a genuine group of Lie type and S be a Sylow p-subgroup.*

(a) *If $S \leq X \leq G$ and $O_p(X) = 1$, then $X = G$.*
(b) *If $X \leq G$ and $O_p(X) \neq 1$, then there is a parabolic P of G, $O_p(P) \neq 1$, such that $X \leq P$.*

Proof. (a) is [GoLyS3, Theorem 2.6.7] and (b) is [GoLyS3, Theorem 3.1.3]. $\square$

Lemma 2.16. [GoLyS3, Theorem 2.5.1.] *Let K be a group of Lie type over $\mathrm{GF}(p^e)$ and $x \in \mathrm{Out}(K)$. Then $x = dfg$ with:*

(a) *d is a diagonal automorphism. In particular $p \nmid o(d)$.*
(b) *f is a field automorphism. In particular if S is a Sylow p-subgroup of K normalized by f, then $X(t)^f = X(t^\sigma)$, where σ is a field automorphism of $\mathrm{GF}(p^e)$ and $X(t)$ is a root group in S. This implies that f also induces a field automorphism on any parabolic containing S and any Levi complement. Recall that twisted groups are not defined over $\mathrm{GF}(p^e)$ but over $\mathrm{GF}(p^{2e})$ or $\mathrm{GF}(p^{3e})$ and σ is an automorphism of this larger field, in particular f might be trivial on Levi factors, which are defined over $\mathrm{GF}(p^e)$.*
(c) *g is a graph automorphism, which comes from a symmetry of the corresponding Dynkin diagram. We have $o(g) = 2$ or 3. The case $o(g) = 3$ just occurs for $K \cong \Omega_8^+(p^e)$. Furthermore $g = 1$, if K is twisted.*

Lemma 2.17. *The groups ${}^2E_6(2)$ and F_2 do not have an involutory automorphism whose centralizer has a component $M(22)$.*

Proof. By the Borel-Tits Theorem 2.15, we see that involutions in ${}^2E_6(2)$ do not have components in their centralizer. By Lemma 2.16 we see that centralizers of outer automorphisms of order 2 have components which are groups of Lie type, so the result follows for ${}^2E_6(2)$. For F_2 it follows directly from [GoLyS3, Table 5.3y]. □

Lemma 2.18. (a) *Let $G \cong L_2(9)$ or $PSp_4(3)$. Then there is no involution i in $\mathrm{Aut}(G)$ such that $C_G(i)$ is an elementary abelian 2-group.*
(b) *If $G \cong PSp_4(3)$ and x is an involution in $\mathrm{Aut}(G)$, then $|C_G(x)|_2 \geq 16$.*

Proof. As $L_2(9) \cong A_6$ the assertion (a) is clear for $G \cong L_2(9)$. As $PSp_4(3) \cong \Omega_6^-(2)$, and $\mathrm{Aut}(G) \cong O_6^-(2)$, we get that

$$C_G(i) \cong 2^{1+4}(Z_3 \times \Sigma_3), 2^4\Sigma_3, \Sigma_6 \text{ or } Z_2 \times \Sigma_4$$

by [AschSe3] and [GoLy, Theorem 9.1]. This is (a).
(b) now follows just by inspection. □

Lemma 2.19. *Let $L = L_4(3)$, $U_4(3)$ or $2U_4(3)$. Then the following holds:*

(i) *If $z \in L \setminus Z(L)$ is a 2-central involution, then $O_2(C_L(z)) \cong Q_8 * Q_8$ or $Z_2 \times Q_8 * Q_8$ in case of $L \cong 2U_4(3)$. Furthermore $C_L(z)/O_2(C_L(z))$ acts faithfully on $O_2(C_L(z))$ and $O_3(C_L(z)/O_2(C_L(z)))$ is elementary abelian of order 9.*
(ii) *$\mathrm{Out}(U_4(3)) \cong D_8$ and $\mathrm{Out}(L_4(3))$ is elementary abelian of order 4.*
(iii) *If $G \cong \mathrm{Aut}(L)$, $L \cong U_4(3)$ and x is an involution in G such that $2^6 \cdot 3^2$ divides $|C_L(x)|$ then one of the following holds:*
  (α) *x is contained in L and 2-central.*
  (β) *$C_L(x) \cong PSp_4(3)$.*
  (γ) *$O_2(C_L(x))$ is elementary abelian, $C_L(x)/O_2(C_L(x))$ acts faithfully on $O_2(C_L(x))$ and induces a group of order 36.*
(iv) *Let $L \cong L_4(3)$ or $U_4(3)$. Then $|Z(T)| = 2$ for T a Sylow 2-subgroup of L. Let G be a subgroup of $\mathrm{Aut}(L)$ containing L and T_1 be a Sylow 2-subgroup of G. If $|\Omega_1(Z(T_1))| > 2$, then $L \cong L_4(3)$ and $|G : L| = 2$. Furthermore some element $t \in \Omega_1(Z(T_1)) \setminus L$ centralizes $PSp_4(3) : 2$ in L.*

Proof. (i) These facts can be found for $U_4(3)$ in [CCNPW, page 52], for $L_4(3)$ in [GoLyS5, Lemma 10.4.15].

(ii) This can be read off from Lemma 2.16.

(iii) Inspection of the orders of centralizers of involutions in $\mathrm{Aut}(L)$ ([CCNPW, page 52]) shows that either (α) or (β) holds or $|C_L(x)| = 2^6 3^2$. In this case also $L\langle x\rangle$ contains an involution y such that $C_L(y) \cong PSp_4(3)$. Let T be a Sylow 2-subgroup of $L\langle y\rangle$, then we have that $J(T)$ is elementary abelian of order 32 and $N_{L\langle y\rangle}(J(T))/J(T) \cong A_6$. This group induces orbits of length 6 and 10 on $J(T) \setminus J(T) \cap L$. Hence we have that x is in the orbit of length 10 and so $C_L(x) \leq N_L(J(T))$. Now $C_{N_L(J(T))}(x)$ is an extension of $J(S) \cap L$ by the normalizer of a Sylow 3-subgroup in A_6. This is (iii).

(iv) We see from [CCNPW, page 52] that in case of $L = U_4(3)$ there is no outer involution x, which will centralize a Sylow 2-subgroup of L, as 2^7 does not divide the order of the centralizer $C_L(x)$. That $|Z(T)| = 2$ follows from (i). For $L = L_4(3)$, we see with [CCNPW, page 68] that there are two classes of outer involutions which centralize $U_4(2) : 2$ in $L_4(3)$. There is a third class corresponding to the diagonal automorphism of order two, which centralizes $L_3(3)$ in L. This shows that the center of a Sylow 2-subgroup of $\mathrm{Aut}(L)$ is of order two. Furthermore in fact there is $G \leq \mathrm{Aut}(L)$ with $|G : L| = 2$, such that $|\Omega_1(Z(T_1))| = 4$ for T_1 a Sylow 2-subgroup of G. □

Lemma 2.20. *If $G = L_4(3), U_4(3), L_3(3)$, $G_2(3)$, $L_2(9)$ or $L_2(p)$, p a prime, and t is some involution in $\mathrm{Aut}(G)$ with nonsolvable centralizer then $G = G_2(3)$ and $C_G(t)$ has a component $L_2(8)$ or $G = U_4(3)$ or $L_4(3)$ and $C_G(t)$ has a component $PSp_4(3)$, $L_3(3)$, $U_3(3)$ or $L_2(9)$.*

Proof. As $L_2(9) \cong A_6$, we easily see in that case that there are no involutions with non solvable centralizer at all. For $L_2(p)$ we get the result with [GoLyS5, Lemma 10.1.3].

For the remaining groups we have by [GoLyS3, Table 4.5.1 - 4.5.3] that centralizers of inner involutions are solvable $\{2, 3\}$–groups or $G \cong L_4(3)$ and we have a component $L_2(9)$. Further we there also find the centralizers of the outer automorphisms. In case of $U_4(3)$ we find a component $U_3(3)$ and in case of $L_4(3)$ we find a component $L_3(3)$, in both cases these are diagonal automorphisms. In both cases $U_4(3)$ and $L_4(3)$ we also find a component $PSp_4(3)$ for a graph automorphism. In case of $G_2(3)$ we get $L_2(8) \cong {}^2G_2(3)'$, which is the centralizer of an outer automorphism. □

We next turn our attention to the groups of Lie type in characteristic two.

Lemma 2.21. [GoLyS3, Theorem 2.5.12.] *Let K be a group of Lie type over* $\mathrm{GF}(2^e)$. *The group of diagonal automorphisms is nontrivial and cyclic exactly in the cases listed below. Its order is given in the second row of the table.*

$L_m(q)$	$U_m(q)$	$E_6(q)$	${}^2E_6(q)$
$(m, q-1)$	$(m, q+1)$	$(3, q-1)$	$(3, q+1)$

Lemma 2.22. *Let G be of Lie type in characteristic two. Let $t \in G$ be an involution. Then $F^*(C_G(t)) = O_2(C_G(t))$.*

Proof. By the the Borel – Tits – Theorem 2.15 we have that $C_G(t)$ is contained in some parabolic P. Now we have that $F^*(P) = O_2(P)$. The $A \times B$–lemma implies that $O_2(P)$ is centralized by any element of odd order in $F^*(C_G(t))$, which is impossible. Hence we get that $F^*(C_G(t)) = O_2(C_G(t))$. ☐

Lemma 2.23. [GoLyS5, Lemma 10.1.2(a)] *Let $G = L_2(q)$, q even. Then there is no $U \leq \mathrm{Out}(G)$ such that $U \cong \Sigma_3$.*

Lemma 2.24. *Let $G = {}^2F_4(q)'$, q even. If $q \neq 2$, then $\mathrm{Out}(G)$ has odd order. If $q = 2$ then $\mathrm{Aut}(G) = {}^2F_4(2)$ and there is no involution in $\mathrm{Aut}(G) \setminus G$.*

Proof. If $q > 2$, this follows from [GoLy, Theorem 9.1]. So assume that $q = 2$. Then with [We] we get $\mathrm{Aut}(G) = {}^2F_4(2)$. Now the assertion follows from [Shi, Corollary 2]. ☐

Lemma 2.25. *Let G be a group and $L = F^*(G)$ be a group of Lie type in characteristic two.*

(1) *If there is an outer automorphism of order 2 of L, which centralizes a Sylow 2–subgroup of L, then $L \cong Sp_4(2)'$.*
(2) *Let t be some outer automorphism of order two of L and K be a component of $C_L(t)$. Then $|Z(K)|$ is odd and K is of Lie type in characteristic two. Furthermore $Z(K) \leq Z(L)$.*
(3) *Assume that L is a central extension of $Sp_{2n}(q)$, $F_4(q)$, ${}^2F_4(q)'$ or $Sz(q)$, $q = 2^n$, and t is an involution in $G \setminus Z(L)$.*
    (i) *If $C_L(t)/O(C_L(t))$ has a component K, then K is a central extension of $Sp_{2n}(s)$, $F_4(s)$, ${}^2F_4(s)'$, $s = 2^b$, or in case of $Sp_4(q)$ also $Sz(q)$ is possible. Further $F^*(L) \not\cong Sz(q)$ or ${}^2F_4(2)$.*
    (ii) *A Sylow 2–subgroup T of $C_L(t)$ is not abelian.*
(4) *Let $L \cong PSL_3(4)$ and $t \in G$ be an involution, which induces an outer automorphism on L. Then $C_L(t) \cong 3^2 : Q_8$, $PSL_2(7)$ or A_5.*

(5) *Assume that L is a central extension of $L_3(q)$ or $U_3(q)$, $q = 2^n$, and let t be an involution in $G\backslash Z(L)$. If $C_L(t)$ has a component K, then K is a central extension of $L_2(s)$, $L_3(s)$ or $U_3(s)$, $s = 2^b$, with $b \leq n$ in the first case and $2b \leq n$ in the remaining cases.*

(6) *Assume that $L \cong G_2(4)$ and there is some involution $t \in G$ such that $C_L(t)$ has a component K, then $K \cong G_2(2)'$.*

Proof. By Lemma 2.22 we have that in (3)(i), (5) and (6) t induces an outer automorphism on L. Hence (1), (2), (3)(i), (5) and (6) follow from [AschSe3, Chapter 19] and Lemma 2.24. We get (4) with [GoLyS5, Lemma 10.2.1].

So we are left with (3)(ii). Suppose false. As the components in (3)(i) do not have abelian Sylow 2-subgroups we have $E(C_L(t)/O(C_L(t))) = 1$. As T is abelian we have that $T \leq C_L(O_{2',2}(C_L(t))/O(C_L(t))) \leq O_{2',2}(C_L(t))$. Hence we get that $TO(C_L(t)) \trianglelefteq C_L(t)$. But then with [AschSe3] we get a contradiction as long as $L \not\cong F_4(q)$ or ${}^2F_4(q)'$. In the latter two cases we just quote [Shi, Corollary 1, Corollary 2]. □

Lemma 2.26. *Let $F^*(G) \in \mathcal{C}_2$ and t be an involution in G centralizing a Sylow 2-subgroup of $F^*(G)$, then either $t \in F^*(G)$ or $F^*(G) \cong A_6$, and t induces the Σ_6–automorphism on $F^*(G)$ or $F^*(G) \cong L_4(3)$ and t induces a graph automorphism on $F^*(G)$.*

Proof. If $F^*(G)$ is a group of Lie type over GF(2) this is Lemma 2.25(1). For $F^*(G)$ sporadic this follows with [GoLyS3, Table 5.3]. For the remaining groups the assertion follows with [GoLy, Theorem 9.1]. □

Next we introduce some notation.

Hypothesis 2.27. Let $G = G(q)$, $q = 2^n$, be a simple group of Lie type, $G \not\cong Sz(q)$, $L_2(q)$ or ${}^2F_4(q)'$. If $G = Sp_{2n}(q)$ let R be a short root group, and a long root group otherwise. Set $X_R = C_G(R)$ and $Q_R = O_2(X_R)$.

Lemma 2.28. *Assume Hypothesis 2.27 with $G \not\cong L_3(q)$, $U_3(q)$, $Sp_4(2)'$ or $G_2(2)'$. Let L be a Levi complement in $N_G(R)$. Then Q_R/R has the following L–module structure:*

(i) *$G \cong L_n(q)$, $O^{2'}(L) \cong SL_{n-2}(q)$, $Q_R/R = V_1 \oplus V_2$, V_1 is the natural L–module and V_2 its dual.*

(ii) *$G \cong \Omega^{\pm}_{2n}(q)$, $O^{2'}(L) \cong \Omega^{\pm}_{2n-4}(q) \times L_2(q) = L_1 \times L_2$, $Q_R/R = V_1 \oplus V_2$, V_i, $i = 1, 2$, are natural L_1–modules and $[Q_R, L_2] = Q_R$.*

(iii) *$G \cong U_n(q)$, $O^{2'}(L) \cong SU_{n-2}(q)$, Q_R/R is the natural module.*

(iv) $G \cong E_6(q)$, $O^{2'}(L) \cong L_6(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{20}$.
(v) $G \cong {}^2E_6(q)$, $O^{2'}(L) \cong U_6(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{20}$.
(vi) $G \cong E_7(q)$, $O^{2'}(L) \cong \Omega_{12}^+(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{32}$.
(vii) $G \cong E_8(q)$, $O^{2'}(L) \cong E_7(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{56}$.
(viii) $G \cong F_4(q)$, $O^{2'}(L) \cong Sp_6(q)$, Q_R/R *is an extension of the natural module by a spin module, where the natural module is contained in* $Z(Q_R)$. *Finally* $Z(Q_R)$ *does not split over* R.
(ix) $G \cong {}^3D_4(q)$, $O^{2'}(L) \cong L_2(q^3)$, Q_R/R *is the* 8*–dimensional* GF(q)*–module for* L.

Proof. This can easily be checked using the Chevalley commutator formula (see also [AschSe3] or [Chapter 3.2][GoLyS3]). In particular in [GoLyS3, Example 3.2.5] one will find the calculation for ${}^3D_4(q)$. For $E_6(q)$, $E_7(q)$, $E_8(q)$, this is [CurKaSei, Proposition 4.4], for $F_4(q)$ we have [CurKaSei, Proposition 4.5]. As in the language of [CurKaSei] the groups G_1 and G_4 are conjugate in $\mathrm{Aut}(G)$, they have the same structure. But in G_4 the $Sp_6(q)$ induces $\Omega_7(q)$ on $Z(O_2(G_4))$, which shows that the same also holds for L and so the module $Z(Q_R)$ does no split. The remaining twisted groups can be found in [CurKaSei, Proposition 4.6].

The classical groups are treated in [CurKaSei, Proposition 3.1 - 3.3] or [GoLyS3, Example 3.2.3]. That the corresponding modules are irreducible is shown in [CurKaSei, Proposition 4.9]. The structure of Q_R and the action of L is also given in the paper [AzBaSei, Theorem 2, Theorem 3]. □

Lemma 2.29. *Let* $K \cong Sp_{2n}(q)$, $n \geq 3$, $q = 2^m$. *We have two root groups* R_1 *and* R_2, *with*

(1) *The Levi factor of* $N_K(R_1)$ *is* $Sp_{2n-2}(q)$, $O_2(N_K(R_1))$ *is elementary abelian and* $O_2(N_K(R_1))/R_1$ *is the natural module.*
(2) *The Levi factor* L *of* $N_K(R_2)$ *is* $Sp_{2n-4}(q) \times L_2(q)$, *furthermore* $Z(O_2(N_K(R_2)))/R_2$ *is the natural* $L_2(q)$*-module, and for* $n > 2$, $O_2(N_K(R_2))' = R_2$, *and* $O_2(N_K(R_2))/Z(O_2(N_K(R_2)))$ *is a tensor product of the two natural modules for the two factors of* L. *If* $q > 2$, *then* $Z(O_2(N_K(R_2)))$ *does not split over* R_2 *as an* $N_K(R_2)$*-module.*

Proof. (1) This is [CurKaSei, Proposition 3.2].

(2) Let V be the natural module for K. Again by [CurKaSei, Proposition 3.2] we have that $N_K(R_2)$ is the stabilizer of an isotropic 2-space W in V, where $W = [V, R_2]$. Now $W \leq W^\perp$ and $N_K(R_2)$ induces on $W^\perp/W$ the corresponding symplectic group $Sp_{2(n-2)}(q)$. Furthermore we have by [CurKaSei, Proposition 3.2] that $C_K(R_1R_2)/O_2(C_K(R_1)) \cong q^{2(n-2)+1}Sp_{2(n-2)}(q)$, which is the centralizer of a long root group in $C_K(R_1)/O_2(C_K(R_1))$. We further see that $C_{O_2(C_K(R_1))}(W)$ has index q in $O_2(C_K(R_1))$. By Witt's result we conclude that $C_K(R_2)$ induces $Sp_2(q)$ on W and so $C_K(R_2)/O_2(C_K(R_2)) \cong Sp_{2(n-2)}(q) \times Sp_2(q)$. Now $O_2(C_K(R_2))$ covers $O_2(C_K(R_1R_2))/O_2(C_K(R_1))$ and so $O_2(C_K(R_2))' \leq R_1R_2$. As some $Sp_2(q)$, which is not contained in $N_K(R_1)$, acts on this commutator group, we see that $O_2(C_K(R_2))' \leq R_2$, with equality for $n \geq 2$. For $n = 2$, we see that $O_2(C_K(R_2))$ is elementary abelian. We furthermore have that $R_1 \leq Z(O_2(C_K(R_2)))$, which, with the action of $Sp_2(q)$, shows that $|Z(O_2(C_K(R_2)))| = q^3$ and modulo R_2, the natural module is induced. Let $n > 2$. From $C_K(R_1)$ we see that $C_K(R_1R_2)/O_2(C_K(R_1R_2))$ induces two $Sp_{2(n-2)}(q)$-modules on $O_2(C_K(R_2))/Z(O_2(C_K(R_2)))$, one in $O_2(C_K(R_1))$ and another one in $O_2(C_K(R_1R_2))/O_2(C_K(R_1))$. As $O_2(C_K(R_1R_2))$ does not act trivially on this group the same applies for the $Sp_2(q)$. Hence we see that $O_2(C_K(R_2))/Z(O_2(C_K(R_2)))$ is a tensor product of the two natural modules.

As $Z(O_2(C_K(R_2)))$ centralizes $W^\perp$, we may embed it into $Sp_4(q)$. Hence it is enough to proof that $Z(O_2(C_K(R_2)))$ does not split over R_2 for $n = 2$. But then we have that R_1 and R_2 are conjugate in $\mathrm{Aut}(K)$ and so we just have to prove that $O_2(C_K(R_1))$ does not split over R_1. Let S be a Sylow 2-subgroup of K such that $R_1R_2 = Z(S)$. Let B be the Borel subgroup, Then $B = SH$, where $H \cong \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$ induces $\mathbb{Z}_{q-1}$ on R_1 and R_2 as well. In particular R_1, R_2 are the only nontrivial H-invariant subgroups in $Z(S)$. As $S' \neq 1$, we get that $S' = R_1R_2$. In particular $R_1 \leq C_K(R_1)'$, which implies that $O_2(C_K(R_1))$ does not split over R_1 as a $N_K(R_1)$-module. $\square$

Lemma 2.30. [DeSte, 10.10 and page 238] *Assume Hypothesis* 2.27 *with* $K \cong G_2(2^e)$, $e \neq 1$. *Let* P *be the normalizer of the root group* R. *Then* $O'(P) \cong (2^e)^{1+4} : SL_2(2^e)$. *If* $e \neq 2$, *then* $O'(P)/Q_R$ *acts irreducibly on* Q_R/R. *If* $e = 2$, *then* P *acts irreducibly on* Q_R/R *but* $O'(P)/Q_R$ *induces a direct sum of two permutation modules for* A_5 *on* Q_R/R.

Let S be a Sylow 2 subgroup of P, then $Z_2(S) \leq Q_R$ and $N_K(Z_2(S))$ induces the natural $L_2(q)$-module on $Z_2(S)$.

Lemma 2.31. [DeSte, 12.9] *Let $K \cong {}^2F_4(q)$, $q = 2^{2n+1}$. Let R be a long root group in K and $P = C_K(R)$. Then $P/O_2(P) \cong Sz(q)$, $R = Z(O_2(P))$, $Z_2(O_2(P))/R$ is an irreducible 4-dimensional module for $P/O_2(P)$, $|C_{O_2(P)}(Z_2(O_2(P)))| = q^6$ and $O_2(P)/C_{O_2(P)}(Z_2(O_2(P)))$ is the natural $P/O_2(P)$-module. Finally $O_2(P)/Z_2(O_2(P))$ is an indecomposable module for $q > 2$.*

If $q = 2$, then $F^*(K) \cong F_4(2)'$ has index 2 in K. We have that $R = Z(O_2(P \cap F^*(K)))$, $Z_2(O_2(P)) = Z_2(O_2(P) \cap F^*(K))$ and $|O_2(P) \cap F^*(K)/Z_2(O_2(P))| = 16$.

In K there is another parabolic P_1 normalizing $Z_2(S)$ and $Z_3(S)$, which is of order q^3. We have that P_1 induces $L_2(q)$ on $Z(O_2(P_1))$.

Lemma 2.32. [GoLyS3, Theorem 3.3.1] *Let K be a group of Lie type in characteristic 2 and S be a Sylow 2-subgroup of K. Then $\Omega_1(Z(S))$ is a root group or $K \cong Sp_{2n}(2^e)$ or $F_4(2^e)$, where $\Omega_1(Z(S))$ is a product of two root groups. In particular if $Z(Q_R) \neq R$, then $G \cong Sp_{2n}(2^e)$ or $F_4(2^e)$.*

Lemma 2.33. *Let $H = F^*(G)$ be one of J_2, $M(24)'$, $L_n(2)$, $\Omega^{\pm}_{2n}(2)$, $U_n(2)$, $E_6(2)$, $E_7(2)$, $E_8(2)$, ${}^2E_6(2)$ or ${}^3D_4(2)$. Let S be a Sylow 2-subgroup of H. Then $\langle r \rangle = Z(S)$ is of order two. If $C_G(r)$ does not act irreducibly on $O_2(C_H(r)/\langle r \rangle)$, then $H = G \cong L_n(2)$, or $H \cong L_3(2)$ or $L_4(2)$.*

Proof. If $F^*(G)$ is a group of Lie type this is Lemma 2.28. For $F^*(G) \cong J_2$ this is [GoLyS3, Table 5.3g] and for $F^*(G) \cong M(24)'$ this is [GoLyS3, Table 5.3v]. □

Lemma 2.34. *Let G be a group such that $F^*(G) = K \in \mathcal{C}_2$ is a simple group. Let S be a Sylow 2–subgroup of G and assume that $|\Omega_1(Z(S))| \geq 4$. If $K \not\in Chev(2)$ then $K \cong L_4(3)$, $L_2(9)$ or $M(23)$.*

Proof. If K is sporadic this follows with [GoLyS3, Table 5.3]. Hence we have that $K \cong L_3(3)$, $U_4(3)$, $L_4(3)$, $G_2(3)$, $PSp_4(3)$, $L_2(9)$ or $L_2(p)$, $p = 2^n \pm 1 > 5$. As by [GoLyS4, Lemma 4.4.2] $\mathrm{Aut}(L_2(p)) = PGL_2(p)$, which by [GoLyS4, Lemma 4.4.1] has a dihedral Sylow 2–subgroup of order $2^n > 4$, we have that $K \ncong L_2(p)$. If $K \cong PSp_4(3)$, $U_4(3)$, $G_2(3)$, $L_3(3)$, we get the assertion with [GoLy, Theorem 9.1]. □

Lemma 2.35. *Let G be one of J_2, $M(24)'$, $\Omega^{\pm}_{2n}(2)$, $n \geq 4$, $E_6(2)$, $E_7(2)$, $E_8(2)$, ${}^2E_6(2)$ or ${}^3D_4(2)$. Let S be a Sylow 2-subgroup of G. Then we have $|Z_2(S)| = 4$.*

Proof. By Lemma 2.33 we have that $|Z(S)| = 2$. Set $Z(S) = \langle r\rangle$. Then again by Lemma 2.33 $C_G(r)$ acts irreducibly on $O_2(C_G(r))/\langle r\rangle$. In all cases but $G \cong M(24)'$ we have that $C_G(r)$ induces an group of Lie type over GF(2) on $O_2(C_G(r))/\langle r\rangle$. By [Sm] we then have that $|C_{O_2(C_G(r))/\langle r\rangle}(S)| = 2$, the assertion.

So assume that $G \cong M(24)'$. Then by [GoLyS3, Table 5.3g]

$$C_G(r)/O_2(C_G(r)) \cong 3U_4(3) : 2,$$

where the normal subgroup of order three is inverted. Furthermore an element of order 3 has to act fixed point freely on $O_2(C_G(r))/\langle r\rangle = W$. Again we will show that $|C_W(S)| = 2$. For this we first take the parabolic subgroup $P = 2^4 3A_6$ in $3U_4(3)$. Then this group induces a faithful $3A_6$–module on $C_W(O_2(P))$. This must be one of the two 6-dimensional module and so a Sylow 2-subgroup T of P centralizes in W a 2-dimensional space. This shows that $|C_W(S\cap P)| = 4$. We have that $Z(P)$ acts faithfully on $C_W(S\cap P)$ and so $\langle S, Z(P)\rangle$ induces $GL_2(2)$ on $C_W(S\cap P)$. Hence $C_W(S)$ is of order two and so $|Z_2(S)| = 4$, the assertion. □

Lemma 2.36. *Assume Hypothesis* 2.27 *with $G \not\cong G_2(2)'$ or A_6. Let H be a hyperplane in $Z(Q_R)$ not containing Q_R', then Q_R/H is extraspecial.*

Proof. As $Q_R' \le Z(Q_R)$, we get that Q_R/H is non abelian with commutator group of order 2. Hence if $C_G(R)$ acts irreducibly on $Q_R/Z(Q_R)$, we have the assertion. So by Lemma 2.28, Lemma 2.29 and Lemma 2.30 we are left with $G \cong L_n(q)$, $n \ge 3$, or $G_2(4)$.

Let first $G \cong L_n(q)$ and E_1, E_2 be the two normal elementary abelian subgroups of order q^{n-1} in Q_R, which correspond to the set of transvection to a point and to a hyperplane on the natural module, respectively. Then G induces $SL_{n-1}(q)$ on these groups and so they are defined over GF(q). Let U be the preimage of $Z(Q_R/H)$. Then $[U, E_i] \le H$, $i = 1, 2$. As $|H| < q$, this implies (recall E_i are modules over GF(q)), that $U \le E_1 \cap E_2 = R$. Hence Q_R/H is extraspecial.

Let $G \cong G_2(4)$ and let U be as before. By Lemma 2.30 we have $Z_2(S) \le Q_R$. In particular $[Z_2(S), U] \le H$. As by Lemma 2.30 $Z_2(S)$ is the natural $L_2(4)$-module, we get that $[Z_2(S), U] = 1$. As Q_R/R is a direct sum of two modules for $C_G(R)$, we see that $Z_2(S)$ intersects both nontrivially and so $Q_R = \langle Z_2(S)^{C_H(R)}\rangle$. This implies $[U, Q_R] = 1$ and so again $U = R$, the assertion. □

Assume Hypothesis 2.27. Then using the lemmas above we get

- $C_G(Q_R) = Z(Q_R)$.
- X_R induces an irreducible module on $Q_R/Z(Q_R)$ for $G \not\cong L_n(q)$ or $G_2(4)$.
- If $Z(Q_R) \neq Q_R$, then $C_{X_R}(Q_R/Z(Q_R)) = Q_R$.
- For $G \cong F_4(q)$ or $Sp_{2n}(q)$ we have that X_R induces on $Z(Q_R)/R$ an irreducible module and an indecomposable module on $Z(Q_R)$ if $G \neq Sp_{2n}(2)$.
- Suppose $G \not\cong Sp_4(q)$. If H is a hyperplane in $Z(Q_R)$ not containing R, then Q_R/H is extraspecial.

Lemma 2.37. *Suppose Hypothesis* 2.27 *with* $G \not\cong G_2(2)'$. *We have* $C_{Q_R}(X_R) = R$.

Proof. Obviously $C_{Q_R}(X_R) \leq Z(Q_R)$ Hence the lemma is true if $R = Z(Q_R)$. So we may assume that $G \cong Sp_{2n}(q)$ or $F_4(q)$. But then by Lemma 2.29 or Lemma 2.28 X_R induces on $Z(Q_R)/R$ an irreducible nontrivial module, the assertion. □

Lemma 2.38. *Suppose Hypothesis* 2.27 *with* $q > 2$. *If* U/Q_R *is normal in* X_R/Q_R, *then* $F^*(U/Q_R)$ *is a product of quasisimple groups, each of which is normal in* X_R/Q_R, *with at most one cyclic group.*

Proof. If $G \cong L_3(q)$ or $U_3(q)$, then Q_R is a Sylow 2-subgroup and X_R/Q_R is a subgroup of the Cartan subgroup, which is a cyclic group in case of $U_3(q)$ and a product of two cyclic groups of order $(q-1)$ and $(q-1)/gcd(3, q-1)$ in case of $L_3(q)$. But the Cartan subgroup induces a cyclic group of order $q-1$ on R, so the assertion holds. In the other cases we have with Lemma 2.28, Lemma 2.29 and Lemma 2.30 that X_R/Q_R is an extension of $O^{2'}(X_R/Q_R)$ by a cyclic subgroup of the Cartan subgroup. Hence all we have to show is that $O^{2'}(X_R/Q_R)$ is a product of quasisimple groups which are normal in X_R. Now the lemmas just quoted show that $O^{2'}(X_R/Q_R)$ is quasisimple besides in the cases $\Omega^{\pm}_{2n}(q)$ and $Sp_{2n}(q)$. If there are just two components, then they cannot be conjugated in X_R/Q_R. So just the case $G = \Omega^+_8(q)$ remains, where we have three components $L_2(q)$. Now $\Omega^+_8(q)$ embeds into $Sp_8(q)$ and so $O^{2'}(X_R)$ embeds into the corresponding group there, which is an extension of a 2-group by $Sp_4(q) \times L_2(q)$ and two of the three components of $O^{2'}(X_R/Q_R)$ embed into the $Sp_4(q)$, in particular they all have to be normal in X_R/Q_R, the assertion. □

Lemma 2.39. *Suppose Hypothesis* 2.27 *with* $q > 2$. *If* U *is a normal subgroup in* X_R *which does not contain* R, *then* $U < R$.

Proof. As $C_G(Q_R) \leq Q_R$ we get that $U \cap Q_R \neq 1$. Suppose first that $U \cap Q_R \not\leq Z(Q_R)$. Let $u \in U \cap Q_R \setminus Z(Q_R)$. Then, as by Lemma 2.36 Q_R/H is extraspecial for any hyperplane H of $Z(Q_R)$ not containing R, we get that $R = [u, Q_R] \leq U$, a contradiction. So we have that $U \cap Q_R \leq Z(Q_R)$. Now $[U, Q_R] \leq Q_R \cap U \leq Z(Q_R)$. If $Q_R \neq Z(Q_R)$, then $C_{X_R}(Q_R/Z(Q_R)) = Q_R$. So we now get that $U \leq Z(Q_R)$. Hence either $U < R$ or $Z(Q_R) \neq R$.

Assume $Z(Q_R) \neq R$. Then either $G \cong Sp_{2n}(q)$ or $G \cong F_4(q)$. In both cases we have by Lemma 2.29 or Lemma 2.28 that X_R acts irreducibly on $Z(Q_R)/R$. This shows that $Z(Q_R) = UR$. But as $q > 2$, we have that X_R induces on $Z(Q_R)$ an indecomposable extension of the trivial module by the natural $L_2(q)$–module, $Sp_6(q)$–module, respectively. As there is a group of order $q - 1$ acting transitively on $R^\sharp$, we see that $R \leq [Z(Q_R), X_R]$ and so $R \leq U$, a contradiction. □

Lemma 2.40. *Let $G = L_3(q)$, $q = 2^n$, and T be a Sylow 2-subgroup of G. Then G possesses two parabolics P_1, P_2 which contain T, such that $U_i = O_2(P_i)$ is elementary abelian of order q^2 and $O^{2'}(P_i/U_i) \cong L_2(q)$, for $i = 1, 2$. Furthermore P_i induces the natural module on U_i, $i = 1, 2$, $T = U_1U_2$ and any involution of T is contained in $U_1 \cup U_2$. Finally there is an automorphism α of G, which normalizes T with $P_1^\alpha = P_2$.*

Proof. We consider $L = SL_3(q)$ instead. Let V be the natural module for L and let P_1 be the point stabilizer. Then $P_1/O_2(P_1) \cong GL_2(q)$ and $U_1 = O_2(P_1)$ is elementary abelian of order q^2, U_1 are just all transvections to this point. Fix T a Sylow 2-subgroup of P_1. By Lemma 2.16 there is a graph automorphism α which normalizes T. Set $P_2 = P_1^\alpha$. Then $U_2 = U_1^\alpha$ is the set of transvections to a hyperplane. In particular $U_1 \neq U_2$. As $N_{P_1}(T) = N_L(T) \leq P_2$ and $N_L(T)$ acts irreducibly on T/U_1, we get that $U_1U_2 = T$. Let $x \in T$ be some involution. Then $[V, x]$ is 1-dimensional. Furthermore $[V, x, x] = 1$, so $[V, x] \leq C_V(x)$. Hence x is a transvection. So all involutions are in $U_1 \cup U_2$. □

Lemma 2.41. *Let $G = L_3(4)$.*

(a) *If T is a Sylow 2-subgroup of G. Then $Z(T)$ is elementary abelian of order 4. Furthermore $C_G(v)$ is solvable for all $1 \neq v \in Z(T)$.*
(b) *Let $H = G\langle\alpha\rangle$, where α is an involution, which induces a graph automorphism on G, then $C_{E(H)}(\alpha) \cong A_5$.*

Proof. (a) By Lemma 2.40 we have that $T = U_1U_2$, where U_1 and U_2 are elementary abelian of order 16 and $C_T(U_1) = U_1$. Hence $Z(T) = U_1 \cap U_2$ is elementary abelian of order 4. If $1 \neq v$ is an involution in $Z(T)$, then

by the Borel-Tits-Theorem 2.15 $C_G(v) \leq P_1$ or P_2 in the language of Lemma 2.40. Now $P_i/O_2(P_i) \cong L_2(4)$, which induces a natural module on $O_2(P_i)$. As $v \in O_2(P_i)$, we have that $C_{P_i}(v)$ is a Sylow 2-subgroup and so solvable.
(b) This is [GoLyS5, Lemma 10.2.1]. $\square$

Lemma 2.42. *Let $X = H\langle\alpha\rangle$, $H/Z(H) \cong L_3(4)$, $Z(H) \leq H'$, where α induces a graph automorphism on. Let Y be a Sylow 2–subgroup of X.*

(a) *$YZ(H)/Z(H)$ does not contain an elementary abelian normal subgroup of order 8.*
(b) *Let $1 \neq Z(H) \leq H'$ be a 2–group. If U is a normal subgroup of Y such that Y/U is abelian or dihedral of order 8, then $Z(H) \cap U \neq 1$.*

Proof. For (a) we may assume $Z(H) = 1$. By Lemma 2.40 we have in H two parabolics P_1, P_2, both extensions of elementary abelian groups of order 16 by $L_2(4)$. Further $P_1^\alpha = P_2$. In $Y \cap H$ we have that all involutions either are in $O_2(P_1)$ or in $O_2(P_2)$. Now let E be elementary abelian of order 8 and E be normal in Y. Then we may assume that $E \cap H \leq O_2(P_1)$. As α does not normalize any subgroup of order 8 in $O_2(P_1)$, we see that $|E \cap H| = 4$ and so $E \not\leq H$. So we have some $e \in E$ with $P_1^e = P_2$. Then $E \cap H = (E \cap H)^e \leq O_2(P_1) \cap O_2(P_2)$ we see that $E \cap H = Z(Y \cap H)$. But then $[E, Y] \leq Z(Y \cap H) \leq O_2(P_1)$ and so E normalizes $O_2(P_1)$, a contradiction. This is (a).

By [Hu, (I.17.4)] we have that $Z(H) \leq (Y \cap H)'$. Let now $U \trianglelefteq Y$. If Y/U is abelian, then $Y' \leq U$ and so $Z(H) \leq U$. So we may assume that Y/U is dihedral of order 8. Assume $Z(H) \cap U = 1$. Then we have that $Z(H)U/U \leq Y'U/U$ is the center of this dihedral group. In particular $|Z(H)| = 2$. As even $Y \cap H/U \cap H$ has to be nonabelian, we may assume that $\alpha \in U$. Now also $[\alpha, Y] \leq U \cap H$. Let W be the preimage of $O_2(P_1)$. As $O_2(P_1)$ is the natural module for P_1, i.e. P_1 acts transitively on the nontrivial elements, we see that W is elementary abelian. Now $Y \cap H = [Y, \alpha]W$ as $Y \cap H = WW^\alpha$. But then we have that $Y \cap H/H \cap U = [Y, \alpha]W/H \cap U \leq (U \cap H)W/H \cap U \cong W/W \cap U$ is elementary abelian, a contradiction. This is (b). $\square$

Lemma 2.43. *Assume Hypothesis 2.27 with $q > 2$. Assume further that $G \not\cong L_3(4)$ and $G \not\cong G_2(4)$. Let $N \leq Q_R$ be a normal subgroup of X_R. If $N \not\leq Z(Q_R)$ then either $N = Q_R$ or $G = L_n(q)$ and N is elementary abelian of order q^{n-1}, or $G = L_3(q)$ and $\Omega_1(N) \leq R$.*

Proof. By Lemma 2.28, Lemma 2.29 and Lemma 2.30 we have that X_R acts irreducibly on $Q_R/Z(Q_R)$ or $G = L_n(q)$. In the case of $L_n(q)$ we have that Q_R/R is a direct sum of the natural $SL_{n-2}(q)$–module by its dual. So N/R is one of the two modules provided $n > 4$.

Assume $G = L_4(q)$. As $q > 2$, we have that $X_R/Q_R \cong L_2(q) \times \mathbb{Z}_{q-1}$. As there is an outer automorphism, the graph automorphism, which inverts the cyclic group of order $q-1$ and centralizes the $L_2(q)$, we see again that the two modules are the only proper X_R–invariant subgroups in Q_R/R and so we get the same result as for $n > 4$.

Finally let $G = L_3(q)$. Then we have a cyclic group Y_R of order $(q-1)/gcd(3, q-1)$ in X_R. As now $q > 4$, we have that Y_R is nontrivial. By Lemma 2.40 $Q_R = E_1E_2$, where the E_i are the two elementary abelian subgroups of order q^2 in Q_R. Further all involutions of Q_R are in $E_1 \cup E_2$. As Y_R acts irreducibly on E_i/R, we see that either $N = E_i$ or $N \cap E_i \leq R$. But then $\Omega_1(N) \leq R$. □

Lemma 2.44. *Suppose Hypothesis* 2.27 *with* $G \cong L_n(q)$, $q = 2^m$, $m \geq 2$, $n \geq 3$, $G \not\cong L_3(4)$. *Then* X_R *has no subgroup of index two.*

Proof. From Lemma 2.28 we see that $O^{2'}(X_R/Q_R) \cong L_{n-2}(q)$. Let first $n \neq 3$. As $q > 2$, $L_{n-2}(q)$ is a simple group. Next we see that X_R induces on $Q_R/Z(Q_R)$ a sum of two modules, which both are nontrivial. If $G \cong L_3(q)$ there is some cyclic group Y_R of order $(q-1)/gcd(3, q-1)$ in X_R. As $q > 4$ in this case, we have that $Y_R \neq 1$. Now in this case $[Q_R, Y_R] = Q_R$. Hence in any case a subgroup U_R of index two in X_R has to cover $O^{2'}(X_R/R)$. As $Z(Q_R) = R$, then $R = O_2(X_R)'$ and so $R \leq U_R$ too, a contradiction. □

Lemma 2.45. *Assume Hypothesis* 2.27 *with* $G \not\cong G_2(2)'$. *Let* t *be a* 2-*element which induces an automorphism of* G *such that* $[t, Q_R] \leq Z(Q_R)$, *then* t *is induced by some element from* Q_R, *or* $G \cong Sp_4(q)'$.

Proof. We may assume that $G \not\cong Sp_4(q)'$. Then we have that $Q_R' = R$ and $Q_R = C_G(Q_R/Z(Q_R))$. This shows that $[t, X_R] \leq Q_R$. By Lemma 2.39 we now get $G \cong L_3(2)$, $L_3(4)$, $L_3(16)$ or $L_4(2)$. In case of $L_4(2)$ the Σ_8–automorphism does not act trivially on $Q_R/Z(Q_R)$. So t is inner. So assume $G \cong L_3(q)$ Then we see that t also has to induces an inner automorphism by [AschSe3, (19.1)] which then has to be in Q_R. □

Lemma 2.46. *Assume Hypothesis* 2.27. *Let* $G \cong Sp_{2n}(q)$, $n \geq 3$, *or* $F_4(q)$, $q = 2^m$, $m \geq 2$. *Let* S *be a Sylow* 2–*subgroup of* G *and* $X = C_G(Z(S))^{(\infty)}$. *If* N *is normal in* X *with* $R \cap N = 1$, *then* $N \leq Z(Q_R)$.

Proof. We have that $Q_R' = R$. Hence $N \cap Q_R \leq Z(Q_R)$ as $N \cap R = 1$ and $N \trianglelefteq X$. As Q_R is normal in X too, we see $[Q_R, N] \leq Q_R \cap N \leq Z(Q_R)$.

But $C_G(Q_R/Z(Q_R)) = Q_R$ and so $N \leq Q_R$, whence $N \leq Z(Q_R)$, the assertion. □

Lemma 2.47. *Let V be a non split extension of a trivial module by the natural module for $X = L_2(q)$, q even. Let S be a Sylow 2-subgroup of X and A be a fours group in S. Then $[V, A] = [V, S]$.*

Proof. Let $\nu \in X$, $o(\nu) = q + 1$ and $\nu^a = \nu^{-1}$ for some $a \in A$. We have that $|[V, \nu]| = q^2$ and so as $|[V, a]| = q$, we see $[V, a] \leq [V, \nu]$. Let $A = \langle a, b\rangle$. We have that $\langle [V, \nu], [V, b]\rangle$ is invariant under $\langle A, \nu\rangle = X$. Hence we conclude that $\langle [V, \nu], [V, b]\rangle = V$ and so $[V, A] = C_V(a) = C_V(S) = [V, S]$. □

Lemma 2.48. *Let $G = Sp_4(q)$, $q = 2^n > 2$, and T be a Sylow 2-subgroup of G. Then G possesses two parabolics P_1, P_2 which contain T, such that $U_i = O_2(P_i)$ is elementary abelian of order q^3 and $P_i/U_i \cong GL_2(q)$, for $i = 1, 2$. We have that U_i is an indecomposable module for P_i and $Z(O^{2'}(P_i)) = R_i$ is a root group. Furthermore $Z(T) = R_1R_2 = T'$, $T = U_1U_2$ and any involution in T is contained in $U_1 \cup U_2$. There is an automorphism α of G with $R_1^\alpha = R_2$ and $P_1^\alpha = P_2$.*

Proof. Let R_1 be the short root group in $Z(T)$ and $P_1 = N_G(R_1)$. The structure of $O^{2'}(P_1)$ is given in Lemma 2.29. Let α be a diagram automorphism, which normalizes T (see Lemma 2.16). Set $P_2 = P_1^\alpha$ and $R_2 = R_1^\alpha$. Then we have that $Z(T) = R_1R_2$. As in $O^{2'}(P_1)$ there is a cyclic group of order $q-1$ which acts transitively on R_1R_2/R_1, we get that this group is in P_2 and so $P_2/U_2 \cong GL_2(q)$. The same applies for P_1 via α. As U_i are indecomposable by Lemma 2.29 we get $R_1R_2 \leq T'$. But in P_1 we see that T/R_1R_2 is abelian, so $T' = R_1R_2 = Z(T)$. As $O^{2'}(P_1/R_1)$ is a split extension of the natural module by $L_2(q)$, we get that T/R_1 is isomorphic to a Sylow 2-subgroup of $L_3(q)$. Now application of Lemma 2.40 gives that any involution of T is contained in $U_1 \cup U_2$. □

Lemma 2.49. *Let $G = Sp_4(q)$, $q = 2^n > 2$, and let T be a Sylow 2-subgroup of G. If α is an outer automorphism of G normalizing T, which is of 2-power order, then $|T : C_T(\alpha)| \geq q^2$.*

Proof. Let $Z(T) = R_1R_2$, $R_i = Z(O'(P_i))$, $i = 1, 2$, in the notation of Lemma 2.48. If $R_1^\alpha = R_2$, then $|Z(T) : C_{Z(T)}(\alpha)| \geq q$. Furthermore $U_1^\alpha = U_2$ and so $|U_1U_2/U_1 \cap U_2 : C_{U_1U_2/U_1\cap U_2}(\alpha)| \geq q$, which gives the assertion.

So we may assume that $R_1^\alpha = R_1$ and $R_2^\alpha = R_2$. By Lemma 2.16 we see that α induces a field automorphism and $q = r^2$. Now $P_1^\alpha = P_1$ and α

induces a field automorphism on $P_1/O_2(P_1)$, which gives that $|T/U_1 : C_{T/U_1}(\alpha)| \geq r$. Furthermore $|Z(T) : C_{Z(T)}(\alpha)| \geq q$. As $[T, U_1] \leq Z(T)$, we have that α induces a field automorphism on $U_1/Z(T)$, which gives $|U_1/Z(T) : C_{U_1/Z(T)}(\alpha)| \geq r$. Together we get $|T : C_T(\alpha)| \geq rqr = q^2$, which proves the lemma. □

Lemma 2.50. *Let $H = Sp_4(q)$, $q = 2^n > 2$, T be a Sylow 2–subgroup of H and R be a root group in $Z(T)$. Then the following hold:*

(i) *If $N \trianglelefteq T$, then $N \ncong \mathbb{Z}_4$ or D_8.*
(ii) *If $N \trianglelefteq T$ with $R \not\leq N$, then $|T : N| \geq q^2/2$. If $|T : N| = q^2/2$, then $|(T/N)'| = 2$. If $|T : N| = q^2$ we have that $|(T/N)'| \leq 4$.*

Proof. By Lemma 2.48 we have that $T = U_1U_2$, where U_i are elementary abelian groups of order q^3 and $\{i \mid i^2 = 1, i \in U\} = U_1 \cup U_2$.

(i) Assume false. Hence we have in both cases that there is some $x \in N$ with $x \notin U_1$. But then $C_{U_1}(x) = Z(T)$, as $N_H(U_1)$ induces $L_2(q)$ on U_1. As by Lemma 2.48 $[U_1, x] \leq Z(T)$ we have that $[U_1, x] \leq \Omega_1(Z(N))$. As $|[U_1, x]| = q$, $q > 2$ and $|\Omega_1(Z(N))| = 2$, this is not possible.

(ii) Suppose $|N : Z(T) \cap N| \geq 8$. Then we may assume that N projects onto $U_1/Z(T)$ with a group of order at least 4. By Lemma 2.47 we get that $N \geq [U_2, N] \geq Z(T)$, a contradiction to $R \not\leq N$.

So we have that $|N : N \cap Z(T)| \leq 4$ and N projects onto each $U_i/Z(T)$ with a group of order at most two. In fact this shows that $|T : N| \geq q^2/2$.

Suppose now that $|T : N| \leq q^2$. In particular we have that $|Z(T) : Z(T) \cap N| \leq 4$. As $T' = Z(T)$ by Lemma 2.48, we get that $(T/N)' \leq Z(T)N/N$ and so $|(T/N)'| \leq 4$. If $|T : N| = q^2/2$, then we have that $|Z(T) : Z(T) \cap N| = 2$ and so $|(T/N)'| = 2$. □

Lemma 2.51. *Let H be a group and $H_1 = F^*(H) \cong Sp_4(q)'$, $q = 2^n$, $U \leq H$ be a 2-group such that*

(i) *$U = \Omega_1(U)$,*
(ii) *$q^4 \geq |U| \geq q^4/2$,*
(iii) *$1 \neq |U'| \leq 4$ and*
(iv) *if $|U| = q^4/2$ then $|U'| = 2$.*

Then $q = 2$.

Proof. Suppose $q > 2$. Let T be a Sylow 2–subgroup of H_1 which is normalized by U. As by Lemma 2.48 $\mathrm{Out}(Sp_4(q))$ has cyclic Sylow 2–subgroups, we have that $|U : U \cap H_1| \leq 2$.

Let R_1, R_2 be the two root groups with $R_1R_2 = Z(T)$. We have that $T = U_1U_2$, $N_{H_1}(U_i) = N_{H_1}(R_i)$, $N_{H_1}(U_i)/U_i \cong GL_2(q)$ and U_i is a non split extension of R_i by the natural module for $GL_2(q)$, $i = 1, 2$ (see Lemma 2.48).

Suppose there is some involution $t \in U$, with $t \notin H_1$. Then by Lemma 2.49

$$(*) \qquad |T : C_T(t)| \geq q^2.$$

We have $|T : U \cap T| \leq 4$, as $|U| \geq |T|/2$. So we see with $(*)$ that $|[t, U]| \geq q^2/|T : U \cap T|$. Hence $8 \geq |U'||T : U \cap T| \geq q^2$, a contradiction. This shows $U \leq H_1$.
As $|T'| = q^2$ by Lemma 2.48, we get that $|U| = q^4/2$ and so $|U'| = 2$. If $x \in T \setminus Z(T)$, then $|[T, x]| \geq q$. Choose $t \in Z(U)$. Then $|[T, t]| \leq |T : U| = 2$. As $q > 2$, we get $t \in Z(T)$. This shows that $Z(U) \leq Z(T)$. Further if $Z(T) \not\leq U$ then $T = Z(T)U$ and then $T' = U'$, which contradicts $|T'| = q^2$. So $Z(T) = Z(U)$. In particular U is a central product of $Z(T)$ with an extraspecial group. Then $|U| = q^2 2^{2n}$ for some n, which contradicts $|U| = q^4/2$. □

Lemma 2.52. *Let $G = G_2(4)$.*

(a) *Let T be a Sylow 2-subgroup of G, then $Z(T)$ is elementary abelian of order 4.*
(b) $\mathrm{Out}(G)$ *induces just field automorphisms on G.*

Proof. (a) This follows from Lemma 2.30.
(b) This follows from Lemma 2.16 and Lemma 2.21. □

Lemma 2.53. *Let $G = L_2(r)$, $L_3(r)$ or $U_3(r)$, $r = 2^f$, and T be a Sylow 2-subgroup of* $\mathrm{Aut}(G)$. *Then $|T| < r^2$ in the first case and $|T| < 2r^4$ in the last two cases.*

Proof. By Lemma 2.16 we have that $|\mathrm{Out}(L)|_2 \leq f$ in the first case and $|\mathrm{Out}(L)| \leq 2f$ in the last two cases. As $f < r$ and $|T| = r$, r^3, respectively, we get the assertion. □

Lemma 2.54. *Let $K \in \mathcal{C}_2$ and $E = SL_2(3) * SL_2(3)$ a subgroup of* $\mathrm{Aut}(K)$, *then $O^2(E)$ induces inner automorphisms.*

Proof. This follows with [GoLyS4, Lemma 4.1.1]. □

In the next definition we sort out some subsets of $\mathcal{C}_2$, which will become important in the proof when we will construct a standard subgroup in the centralizer of a 2-central involution.

Definition 2.55. (a) By $\mathcal{M}$ we denote the set

$$\{U_6(2), Sp_6(2), \Omega_8^+(2), F_4(2), Sz(8), G_2(4), L_3(4), {}^2E_6(2),\\ M(22), F_2, M_{22}, M_{12}, Suz, Co_1, J_2, Ru, HiS, U_4(3)\}.$$

(b) By $\mathcal{M}_1$ we denote the set

$$\{M(22), 2M(22), M(23), M(24)', F_2, 2F_2, F_1, 2Suz, 2Ru, 2Co_1,\\ 2\Omega_8^+(2), 2^2\Omega_8^+(2), 2U_6(2), 2^2U_6(2), 2\cdot{}^2E_6(2), 2^2\cdot{}^2E_6(2)\}.$$

(c) By $\mathcal{M}_2$ we denote the set

$$\{M(22), 2M(22), M(23), M(24)', F_2, 2F_2, F_1\}.$$

(d) By $\mathcal{R}$ we denote the set

$$\{U_6(2), Sp_6(2), \Omega_8^+(2), {}^2F_4(2)', F_4(2),\\ {}^2E_6(2), M(22), M(23), M(24)', F_2, F_1\}.$$

The groups in $\mathcal{M}$ are exactly those in $\mathcal{C}_2$, whose nontrivial central extensions by 2-groups are again in $\mathcal{C}_2$ (see Lemma 2.56). The groups in $\mathcal{M}_1$ are the groups $G \in \mathcal{C}_2$ which possess an involution in $\mathrm{Aut}(G)$ such hat $C_G(t)$ has a component $K \in \mathcal{C}_2$ with $Z(K) \neq 1$, and the groups in $\mathcal{M}_2$ are in $\mathcal{M}_1$ but $Z(K) \not\leq Z(G)$ (see Lemma 2.60). The meaning of the set $\mathcal{R}$ will become clear in Chapter 4. There we introduce a certain relation on components of involution centralizer.

The set $\mathcal{R}$ consists of those terminal elements, which cannot be reached from elements outside of $\mathcal{R}$ (see Lemma 4.12 and Lemma 4.13).

Lemma 2.56. *Let $G \in \mathcal{C}_2$ with $Z(G) \neq 1$, then $G/Z(G) \in \mathcal{M}$.*

Proof. This can be found in [GoLyS3, Table 5.3] for the sporadic groups and [GoLyS3, Theorem 6.1.4] for the groups of Lie type. □

Lemma 2.57. *If $G \in \mathcal{C}_2$ and $G/Z(G)$ has abelian Sylow 2–subgroups, then $Z(G) = 1$.*

Proof. Suppose false, then by Lemma 2.56 $G/Z(G) \in \mathcal{M}$. But there are no groups with abelian Sylow 2–subgroup in $\mathcal{M}$. □

We will quite often use that groups cannot be involved in other groups. For this we sometimes just use the orders. We now collect this information.

Lemma 2.58. *For $L \in \mathcal{M}$ the order of L is given below:*

$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$
$Sp_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$
$\Omega_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$
$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$
$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
${}^2E_6(2)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$
$M(22)$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
F_2	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
Ru	$2^{14} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 29$
HiS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$

Proof. For the groups of Lie type this can be read of from [GoLyS3, Table 2.2]. For the sporadic groups it follows with [GoLyS3, Table 5.3]. □

Lemma 2.59. *The Schur multipliers for the groups $L \in \mathcal{R}$ are as follows:*

$U_6(2)$	$2^2 \times 3$	$M(22)$	6
$Sp_6(2)$	2	$M(23)$	1
$\Omega_8^+(2)$	2^2	$M(24)'$	3
${}^2E_6(2)$	$2^2 \times 3$	F_2	2
$F_4(2)$	2	F_1	1
${}^2F_4(2)'$	1		

Proof. This is [GoLyS3, Theorem 6.1.2] and [GoLyS3, Definition 6.1.3]. □

Lemma 2.60. *Let $G \in \mathcal{C}_2$, $t \in \mathrm{Aut}(G)$ be an involution such that $C_G(t)$ has a component $K \in \mathcal{C}_2$ with $Z(K) \neq 1$.*

(a) *We have that $G \in \mathcal{M}_1$.*
(b) *If $Z(K) \not\leq Z(G)$, then $G \in \mathcal{M}_2$.*
(c) *If $|Z(K)| \geq 4$, then $Z(K) \not\leq Z(G)$.*

Proof. By Lemma 2.56 $K/Z(K) \in \mathcal{M}$. Suppose first that $G/Z(G)$ is a group of Lie type in characteristic two. Then by Lemma 2.22 t is an outer automorphism and components have center of odd order. Hence $Z(K) \leq Z(G)$. In particular $Z(G) \neq 1$ and so by Lemma 2.56 $G/Z(G) \in \mathcal{M}$. Hence we have (a) if we can show that $G/Z(G) \not\cong Sp_6(2)$, $Sz(8)$, $L_3(4)$, $G_2(4)$ or $F_4(2)$. The first two groups do not have an involutory outer automorphism. As $G \in \mathcal{C}_2$ and $Z(G) \neq 1$, we now get $G \cong 2L_3(4)$, $2^2L_3(4)$, $2G_2(4)$ or $2F_4(2)$. But $C_G(t)$ is nonsolvable and so by Lemma 2.25 we have that $K/Z(K)$ is of Lie type in characteristic two, which then is $L_2(4)$, $L_3(2)$, $G_2(2)'$ or ${}^2F_4(2)'$. But then $K/Z(K) \notin \mathcal{M}$, a contradiction. Further in all cases we see that for $G/Z(G)$ a group of Lie type in characteristic two we also have (b).

Let next $G/Z(G)$ be sporadic. If $Z(K) \leq Z(G)$, we again get that $G/Z(G) \in \mathcal{M}$. Hence we have proved (a) if we can show that $G/Z(G) \not\cong M_{22}$, M_{12}, J_2 or HiS. But in these cases by [GoLyS3, Table 5.3] none of the components of $C_G(t)$ are in $\mathcal{M}$. So we have that $Z(K) \not\leq Z(G)$. Then we have a nonsimple component in a sporadic group and so by [GoLyS3, Table 5.3] $G/Z(G) \cong M(22)$, $M(23)$, $M(24)'$, F_2 or F_1. This proves (a) and (b).

So we are left with $G/Z(G) \cong L_3(3)$, $G_2(3)$, $L_4(3)$, $U_4(3)$, $L_2(9)$ or $L_2(p)$. Now we may apply Lemma 2.20. So we get $K/Z(K) \cong PSp_4(3)$, $L_3(3)$, $U_3(3)$, $L_2(9)$ or $L_2(8)$, which all are not in $\mathcal{M}$.

To prove (c) assume that $|Z(K)| \geq 4$ and $Z(K) \leq Z(G)$. Then by (a) $G \in \mathcal{M}_1$. In particular $G \cong 2^2\Omega_8^+(2)$, $2^2U_6(2)$ or $2^2\cdot\, {}^2E_6(2)$. By [GoLy, Theorem 9.1] the possible components $K/Z(K)$ are $Sp_6(2)$ and $F_4(2)$. But then by Lemma 2.59 $Z(K)$ is of order two, a contradiction. This proves (c). $\square$

Lemma 2.61. *Let $G \in \mathcal{C}_2$. If some $K \in \mathcal{M}_1$ is a component of the centralizer of some involution in G, then $G/Z(G) \in \mathcal{M}_2$.*

Proof. By Lemma 2.22 and Lemma 2.20 we have that $G/Z(G)$ must be sporadic. Now the assertion follows with [GoLyS3, Table 5.3]. $\square$

Lemma 2.62. *If $K \in \mathcal{M}_2$ and $G \in \mathcal{C}_2$ such that $G/N \cong K$ for some $N \leq Z(G)$, then $G \in \mathcal{M}_2$.*

Proof. This of course is true if $N = 1$. So assume that $N \neq 1$. Then by definition of $\mathcal{C}_2$ we have that N is in the Schur multiplier of K and $|N|$ is a power of 2. As the groups in $\mathcal{M}_2$ are all sporadic and according to

[GoLyS3, Table 5.3] for every group in $\mathcal{M}_2$ also any Schur extension by a 2-group is contained in $\mathcal{M}_2$, the assertion follows. □

Lemma 2.63. *Let $G \in \mathcal{C}_2$. If $G/Z(G)$ is group of Lie type over* $\mathrm{GF}(q)$*, $q = 2^n$, $n \geq 2$, such that $1 \neq Z(G) \leq G'$. Then $G/Z(G) \cong L_3(4)$, $G_2(4)$ or $Sz(8)$.*

Proof. This follows with Lemma 2.56. □

Lemma 2.64. *If $G \in \mathcal{M}_1$ and $t \in$* $\mathrm{Aut}(G)$ *is an involution, then $C_{G/Z(G)}(t)$ has at most one component.*

Proof. If $G/Z(G)$ is sporadic this follows with [GoLyS3, Table 5.3]. So assume now that $G/Z(G)$ is of Lie type in characteristic two. Then by Lemma 2.22 we have that $t \notin G$. Now with [GoLy, Theorem 9.1] we get the assertion. □

Lemma 2.65. *Let K, K_1, L be in $\mathcal{M}_1$. Suppose there is $1 \neq N \leq Z(K_1)$ with $K_1/N = K$. Assume furthermore that for some noncentral involution $t \in$* $\mathrm{Aut}(L)$ *we have that $C_L(t)$ has a component K_1. Then $L \in \mathcal{M}_2$.*

Proof. We have that K and K_1 are described in Lemma 2.60. Furthermore both K and a nontrivial central extension of K has to be in $\mathcal{M}_1$. Inspection of the groups in Lemma 2.60 shows that $K_1 \cong 2^2\ {}^2E_6(2)$, $2^2U_6(2)$, $2^2\Omega_8^+(2)$, $2M(22)$ or $2F_2$. With Lemma 2.60 we get the assertion except when $|Z(K_1)| = 2$ and $Z(K_1) \leq Z(L)$. In particular $K_1 \cong 2M(22)$ or $2F_2$. As no group $L \in \mathcal{M}_1$ with $Z(L) \neq 1$ apart from $2F_2$ has an order divisible by 2^{42} (see Lemma 2.58), we conclude that $K_1 \ncong 2F_2$. The same order argument shows that $2M(22)$ could only be contained in $2Co_1$ or $2 \cdot {}^2E_6(2)$. But by Lemma 2.17 there are no involutions in the automorphism groups of these groups which centralize a component $M(22)$. □

Lemma 2.66. *Let $G \in \mathcal{R}$. If t is some involutory automorphism G, then t centralizes elements of odd order in G.*

Proof. If G is sporadic, we find the assertion in [GoLyS3, Table 5.3]. If G is of Lie type in characteristic two and t is some outer automorphism this is [GoLy, Theorem 9.1]. If it is an inner automorphism this is [AschSe3]. □

Lemma 2.67. *(a) If $L \in \mathcal{R}$ and T is a Sylow 2-subgroup of L with $|Z(T)| > 2$, then $L \cong M(23)$ $F_4(2)$ or $Sp_6(2)$.*
(b) The class of a Sylow 2-subgroup for $L \in \mathcal{R}$ is at least three.

Proof. (a) Let T be a Sylow 2-subgroup of L. If T contains an extraspecial subgroup Q with $C_T(Q) = Z(Q)$, we have that $|Z(T)| = 2$. By Lemma 2.28 this is true for $L \cong U_6(2)$, $\Omega_8^+(2)$ and ${}^2E_6(2)$. By [GoLyS3, Table 5.3 v,y,z] this is also true for $L \cong M(24)'$, F_2 and F_1. So we are left with $L \cong M(22)$ and ${}^2F_4(2)$. In the latter the assertion follows with Lemma 2.31. If $L \cong M(22)$ there is a non 2-central involution x in L such that $2U_6(2) \cong C_L(x)$. Application of [GoLyS3, Table 5.3t] shows that the center of a Sylow 2-subgoup of $C_L(x)$ is of order 4. As x is not 2-central we have that $|Z(T)| = 2$.

(b) Suppose that T is of class at most two. Then $T/Z(T)$ is abelian. But now using Lemma 2.28 and Lemma 2.31, we see that this is absurd for the groups of Lie type in $\mathcal{R}$. As the class of a Sylow 2-subgroup of $U_6(2)$ now is at least three and $2U_6(2) \leq M(22) \leq M(23) \leq M(24)'$ and $2U_6(2)$ is also involved in F_2, which is involved in F_1, we get the assertion also for the sporadic groups in $\mathcal{R}$. □

3. Examples

In this chapter we show under what circumstances the examples $\Omega_7(3)$, $\Omega_8^-(3)$ and A_{12} in Theorem 1.4 arise.

Lemma 3.1. *If G is a group of Lie type in odd characteristic which is of even type but not of even characteristic, then $G \cong \Omega_7(3)$ or $\Omega_8^-(3)$.*

Proof. The structure of centralizers of involutions in G as far as needed in this proof can be found in [GoLyS3, Chapter 4.5]. By assumption $G \not\cong L_2(q)$, as G is of even type. If G is a Ree group, then $C_G(z) = \langle z \rangle \times L_2(q)$, were $q = 3^{2n+1}, n \geq 1$, for all involutions z. As components are in $\mathcal{C}_2$, this is not possible. In the remaining cases we have some $SL_2(q)$ subnormal in $C_G(z)$ for some involution z. As there are no components $SL_2(q)$, as $SL_2(q) \notin \mathcal{C}_2$, this shows $q = 3$. Now from [GoLyS3, Chapter 4.5] we get the following list:

Group		Component in some involutions centralizer	
$L_n(3)$,	$n \geq 5$	$SL_{n-2}(3)$,	n even
		$SL_{n-1}(3)$,	n odd
$Sp_{2n}(3)$,	$n \geq 3$	$Sp_{2n-2}(3)$	
$U_n(3)$,	$n \geq 5$	$SU_{n-2}(3)$,	n even
		$SU_{n-1}(3)$,	n odd
$\Omega_n^{\pm}(3)$,	$n \geq 11$	$\Omega_{n-4}^{\pm}(3)$	
$\Omega_{10}^{\pm}(3)$		$\Omega_8^{\pm}(3)$	
$\Omega_9(3)$		$\Omega_7(3)$	
${}^3D_4(3)$		$SL_2(27)$	
$F_4(3)$		$Sp_6(3)$	
${}^2E_6(3)$		$SU_6(3)$	
$E_6(3)$		$SL_6(3)$	
$E_7(3)$		$SO_{12}(3)$	
$E_8(3)$		$E_7(3)$	

We see that none of these components are in $\mathcal{C}_2$. Assume now that $G \not\cong \Omega_7(3)$ or $\Omega_8^-(3)$. Then what is left are the groups $G = L_4(3)$, $L_3(3)$, $PSp_4(3)$, $U_4(3)$, $U_3(3)$, $\Omega_8^+(3)$ and $G_2(3)$. In all these groups the centralizer of any 2-central involution is solvable and so these groups are of even characteristic, hence they do not satisfy the assumption of this lemma. This proves the lemma. □

A group X of even order is called tightly embedded in G if $|X \cap X^g|$ is even implies $X^g = X$.

Lemma 3.2. *Let G be a simple group of of even type which is not of even characteristic. Assume that G possesses a subgroup X such that one of the following holds:*

(1) *X is a tightly embedded quaternion subgroup or;*
(2) *$X \cong SL_2(3)$ and is subnormal in $C_G(Z(X))$. Furthermore for any $g \in C_G(Z(X))$ we either have $X^g \cap X = Z(X)$ or $X^g = X$.*

Then $G \cong \Omega_7(3)$ or $\Omega_8^-(3)$.

Proof. By [Asch2] the simple groups G, which satisfy (1) or (2) are M_{11}, M_{12} or a group of Lie type in odd characteristic. As M_{11} and M_{12} both do not satisfy the assumptions of this lemma, we see that G is of Lie type in odd characteristic. Now Lemma 3.1 proves the assertion. □

Definition 3.3. [GoLy, page 133] Let G be a group. A subgroup $L \in \mathcal{L}$ of G is called standard, if

(1) $C_G(L)$ is tightly embedded in G and
(2) $[L, L^g] \neq 1$ for all $g \in G$.

Lemma 3.4. *Let G be a group of even type, which is not of even characteristic. Let $L \in \mathcal{C}_2$ be a standard subgroup of G. If $G \not\cong \Omega_7(3)$ or $\Omega_8^-(3)$, then $C_G(L)$ possesses an abelian Sylow 2–subgroup U. If $m_2(U) > 1$, then U is elementary abelian.*

Proof. If $m_2(U) > 1$, the assertion follows from [Asch1, Theorem 4]. So assume $m_2(U) = 1$ and U is quaternion. As there are no components $K \in \mathcal{C}_2$ with $m_2(K) = 1$, we see that $C_G(L)$ is solvable. Further $O(C_G(Z(U))) = 1$. But then either $C_{C_G(L)}(Z(U)) = U$ or there is a subgroup $U_1 \cong SL_2(3)$ which is normal in $C_{C_G(L)}(Z(U))$. But both cases are not possible by Lemma 3.2. Recall that in the first case U is tightly embedded in G. □

Proposition 3.5. *Let G be of even type but not of even characteristic. Let L be a standard subgroup of G such that $N_G(L)$ contains a Sylow 2–subgroup of G. If $C_G(L)$ has elementary abelian noncyclic Sylow 2–subgroups, then $G \cong A_{12}$.*

Proof. Let U be a Sylow 2–subgroup of $C_G(L)$ and T_1 be a Sylow 2–subgroup of L. Assume first that L is some Bender group $B(q)$. Then $E = U \times \Omega_1(T_1)$ is an elementary abelian subgroup of order $q|U|$. By [Asch1, Theorem 2] there is $U^g \leq N_G(L)$ with $U^g \cap C_G(L) = 1$. In particular as a Sylow 2-subgroup of $\mathrm{Out}(L)$ is cyclic and U is not, there is some $u \in U^\sharp$ such that $u^g \in E \setminus U$. We are going to apply Lemma 2.6 to E. For this we have to show that $(N_G(E)/C_G(E), E)$ is a Goldschmidt pair. We set $V_1 = \Omega_1(T_1)$. Then in L there is a cyclic group acting transitively on $V_1^\sharp$. Furthermore for $V_0 = U$, conditions (iii) and (iv) of Definition 2.5 hold. As $N_G(L)$ contains a Sylow 2–subgroup of G, we see that the number of conjugates of U under $N_G(E)$ is odd. As $|U| > 2$, we have that Lemma 2.6(iv) cannot hold, so with Lemma 2.6 we see that U is normal in $N_G(E)$, in particular $g \notin N_G(E)$. Hence E cannot be the only elementary abelian subgroup of its order in S, S a Sylow 2–subgroup of $N_G(L)$ containing E. Furthermore there must be at least three of them in S. This gives that $L \cong U_3(q)$ and some involutory field automorphism is induced on L. But $[T_1, u^g] = 1$, and E is the only elementary abelian subgroup of S of order $|E|$, which centralizes a special group of order q^3. Hence again $g \in N_G(E)$, a contradiction.

Assume now that $L \cong L_3(2^n)$. Then [AschSe1, (6.1)] shows $q = 4$. Furthermore [AschSe2, (1.5)] gives that $N_G(L)$ does not contain a Sylow 2-subgroup. If $L \cong G_2(4)$ we get with [AschSe2, (1.6)] that $N_G(L)$ does not contain a Sylow 2–subgroup. For all the other cases of a group of Lie type in characteristic two, which is not alternating at the same time, we get a contradiction with [AschSe1, (7.1)]. If L is of Lie type in odd characteristic but not alternating, we get a contradiction with [AschSe1, (4.9)]. If L is sporadic then [AschSe1, (17.1)] gives the contradiction. Hence we are left with L alternating. This shows that $L/Z(L) \cong A_n$. As $L \in \mathcal{C}_2$, we get $L \cong A_5$, A_6, A_8. Now we can quote [Asch3] which shows that $G \cong J_2$, M_{12}, A_9, A_{10} or A_{12}. As G is not of even characteristic, we have $G \cong A_{12}$. $\square$

4. The standard subgroup

In this chapter we fix a Sylow 2–subgroup S of G and assume that G is of even type but not of even characteristic. This means by Lemma 2.1 that there is some $1 \neq z \in Z(S)$, $z^2 = 1$, such that $C_G(z)$ possesses a component $A_z \in \mathcal{C}_2$. Further we assume that G is a counterexample to Theorem 1.4. This means that $G \ncong \Omega_7(3)$, $\Omega_8^-(3)$ or A_{12}. In particular by Lemma 3.2 G does not possess a tightly embedded quaternion subgroup or a subgroup $X \cong SL_2(3)$, which is subnormal in $C_G(Z(X))$ such that for any $g \in C_G(Z(X))$ we either have $X^g \cap X = Z(X)$ or $X^g = X$. By Proposition 3.5 we also have that there is no standard subgroup L such that $N_G(L)$ contains a Sylow 2-subgroup of G and $C_G(L)$ has noncyclic Sylow 2-subgroups. The aim of this chapter is to prove:

Proposition 4.1. *There exists an involution $z \in Z(S)$ whose centralizer has a component $A_z \in \mathcal{C}_2$, such that A_z is standard and $C_G(A_z)$ has a cyclic Sylow 2–subgroup.*

By $\mathcal{L}$ we denote the set of all components of the involution centralizers of G. We first describe a procedure which will provide us with a standard subgroup in a centralizer of some involution in G.

Definition 4.2. Let $K, L \in \mathcal{L}$. We say $K \sqsubseteq L$ if $K = L$ or L is a component of $C_G(u)$ for some involution u and there is an involution $t \in C_G(u)$ such that $[L, t] = L$ and K is a component of $C_L(t)$. Let $\leq^*$ be the transitive extension of $\sqsubseteq$.

This partial order $\leq^*$ was investigated by M. Aschbacher in [Asch1].

Definition 4.3. Let $K \in \mathcal{L}$.

(a) By $\mathcal{L}_K^*$ we denote the set of maximal elements in $\mathcal{L}$ with respect to $\leq^*$, which contain K.

(b) By $\mathcal{K}_K$ we denote the set of $L \in \mathcal{L}$ with $L \notin \mathcal{L}_L^*$, such that $L/N \cong K$ for some $N \leq Z(L)$.

Definition 4.4. Let $K, L \in \mathcal{L}$.

(a) We write $K \to L$ if there is some chain

$$K = K_1, K_2, \dots, K_r = L, K_i \in \mathcal{L} \text{ for all } i,$$

such that either $K_i \in \mathcal{L}_{K_{i-1}}^*$ or $K_{i-1} \in \mathcal{L}_{K_{i-1}}^*$ and $K_i \in \mathcal{K}_{K_{i-1}}$. We set further

$$\bar{\mathcal{L}}_K = \{L \mid L \in \mathcal{L}, K \to L\}.$$

(b) We set

$$\bar{\mathcal{L}}_K^* = \{L \mid L \in \bar{\mathcal{L}}_K \text{ with } \bar{\mathcal{L}}_L = \{L\}\}.$$

From now on components of centralizers of involutions are always in $\mathcal{C}_2$. As $\mathcal{C}_2$ contains no elements K with $m_2(K) = 1$, we get from [Asch1, Theorem 1]:

Proposition 4.5. *If $L \in \mathcal{C}_2$ and $L \in \bar{\mathcal{L}}_K^*$, then L is a standard subgroup.*

Our aim is to produce a standard subgroup which is normalized by a Sylow 2-subgroup of G. But even if we start with a component $K \in \mathcal{L}$ which is normalized by a Sylow 2-subgroup there is no reason why the standard subgroup $L \in \bar{\mathcal{L}}_K^*$ we get using Proposition 4.5 should have this property too. To get control over this standard subgroup we have to study the procedure more carefully. In particular we need information about the penultimate group in the construction. This will be done in the next lemmas.

Lemma 4.6. *Let $K \in \mathcal{C}_2$ and $L \in \bar{\mathcal{L}}_K^*$. Then*

(a) $|K|\big||L|$;

(b) *if $K = K_1, \dots, K_r = L$ is a chain for $K \to L$, then $L \notin \mathcal{K}_{K_{r-1}}$.*

Proof. Let $K = K_1, \dots, K_r = L$ be a chain as in (b). Let i be such that $|K|\big||K_i|$. Assume $i < r$. Now $K_{i+1} \in \mathcal{L}_{K_i}^*$ or $K_{i+1} \in \mathcal{K}_{K_i}$. In the first case $K_i \leq K_{i+1}$ and in the second case $K_{i+1}/N \cong K_i$ for some $N \leq Z(K_{i+1})$. Hence in both cases $|K_i|\big||K_{i+1}|$ and so $|K|\big||K_{i+1}|$. By induction we get (a).

For (b) assume $L \in \mathcal{K}_{K_{r-1}}$. Then $L \notin \mathcal{L}_L^*$. In particular $\bar{\mathcal{L}}_L \neq \{L\}$. But then $L \notin \bar{\mathcal{L}}_K^*$, a contradiction. □

Lemma 4.7. *Let $K \in \mathcal{C}_2$, $Z(K) \neq 1$. If $L \in \mathcal{C}_2$ with $K \sqsubseteq L$ and $K \neq L$, then $L \in \mathcal{M}_1$.*

Proof. By Lemma 2.56 $K/Z(K) \in \mathcal{M}$. By Definition 4.2 there is some involution t such that $[L,t] = L$ and K is a component of $C_L(t)$. Then the assertion follows with Lemma 2.60. $\square$

Lemma 4.8. *If $K \in \mathcal{M}_2$ and $K \sqsubseteq L$, then $L \in \mathcal{M}_2$. Further $\bar{\mathcal{L}}^*_K \subseteq \mathcal{M}_2$.*

Proof. If $K = L$ we are done. So we may assume that $K \sqsubseteq L$, $L \in \mathcal{C}_2$, with $K \neq L$. Then $C_L(t)$ has a component K for some involution t in $N_G(L)$. By Lemma 2.61 $L \in \mathcal{M}_2$. As $\mathcal{M}_2$ is closed under even Schur multipliers by Lemma 2.62, we get that $K \to L$ implies $L \in \mathcal{M}_2$, the assertion. $\square$

Lemma 4.9. *If $K/Z(K) \in \mathcal{M}$ with $Z(K) \neq 1$ and $L \in \mathcal{L}^*_K$, $L \neq K$, then $L \in \mathcal{M}_1$.*

Proof. Let $K \sqsubseteq L_1 \sqsubseteq L_2 \cdots \sqsubseteq L$. By Lemma 4.7 and induction on the length of a chain we may assume that $Z(L_1) = 1$. Hence L_1 possesses an involutory automorphism t such that $C_{L_1}(t)$ has a component K. Now application of Lemma 2.60 (b) implies $L_1 \cong M(22)$, $M(23)$, $M(24)'$, F_2 or F_1. So $L_1 \in \mathcal{M}_2$. Then by Lemma 4.8 we have $L \in \mathcal{L}^*_{L_1} \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_1$, the assertion. $\square$

Lemma 4.10. *If $K \in \mathcal{M}_1$ and $K \notin \bar{\mathcal{L}}^*_K$, then $\bar{\mathcal{L}}^*_K \subseteq \mathcal{M}_2$.*

Proof. Let $L \in \bar{\mathcal{L}}^*_K$. Hence $K \to L$ with $K = K_1, K_2, \ldots, K_r = L$ the corresponding chain. By assumption $L \neq K$. So we may assume that $K \neq K_2$. Suppose first that $K_2 \in \mathcal{L}^*_K$. Then we have $K \sqsubseteq L_1 \sqsubseteq L_2 \cdots \sqsubseteq K_2$. We may assume $L_1 \neq K$. By Lemma 2.61 $L_1 \in \mathcal{M}_2$. As $L \in \bar{\mathcal{L}}^*_{L_1}$ the assertion follows with Lemma 4.8.

So we may assume $K \notin \mathcal{L}^*_K$ and $K_2 \in \mathcal{K}_K$. Hence there is $1 \neq N \leq Z(K_2)$ such that $K_2/N \cong K$. By Lemma 4.7 we have $K_2 \in \mathcal{M}_1$. By definition $K_2 \notin \mathcal{L}^*_{K_2}$, in particular there is $K_3 \in \mathcal{L}^*_{K_2}$. Now let again $K_2 \sqsubseteq L_1 \sqsubseteq L_2 \cdots \sqsubseteq K_3$ and $K_2 \neq L_1$. Then application of Lemma 2.65 yields $L_1 \in \mathcal{M}_2$. As $L \in \bar{\mathcal{L}}^*_{L_1}$ the assertion follows with Lemma 4.8. $\square$

Lemma 4.11. *If $K/Z(K) \in \mathcal{M}$, $Z(K) \neq 1$ and $K \in \mathcal{L}^*_K \setminus \bar{\mathcal{L}}^*_K$, then $\bar{\mathcal{L}}^*_K \subseteq \mathcal{M}_2$.*

Proof. Let $L \in \bar{\mathcal{L}}^*_K$ and $K = K_1, K_2, \ldots, K_r = L$ be a chain corresponding to $K \to L$. As $K \in \mathcal{L}^*_K$, we have that $K_2 \in \mathcal{K}_K$. So $K_2/N \cong K$ for some $1 \neq N \leq Z(K_2)$. Now $|Z(K_2)| \geq 4$. As $K_2 \notin \bar{\mathcal{L}}^*_{K_2}$ by definition of $\mathcal{K}_K$ (Definition 4.3) we get that $K_2 \sqsubseteq L_1$. By Lemma 2.60 we see $L_1 \in \mathcal{M}_1$ and as $|Z(K_2)| \geq 4$, we see with Lemma 2.60(c) and (b) that $L_1 \in \mathcal{M}_2$. As $\bar{\mathcal{L}}^*_K = \bar{\mathcal{L}}^*_{L_1}$ we get the assertion with Lemma 4.10. $\square$

We will see that the elements in $\mathcal{R}$ are terminal elements in our partial order $\to$, which cannot be reached from elements not in $\mathcal{R}$. For the definition of $\mathcal{R}$ see Definition 2.55.

Lemma 4.12. *Let $L \in \mathcal{C}_2$ with $L/Z(L) \in \mathcal{R}$ and $K \in \mathcal{C}_2$ with $K \sqsubseteq L$, then $K/Z(K) \in \mathcal{R}$. In particular if $L \in \mathcal{L}_K^*$ for some $K \in \mathcal{C}_2$, then $K/Z(K) \in \mathcal{R}$.*

Proof. If $L = K$ we have nothing to prove. Otherwise there is some involution t with $L = [L, t]$ such that K is a component of $C_L(t)$. Now the assertion follows with [GoLyS3, Table 5.3] for the sporadic groups and [GoLy, Theorem 9.1] for the groups of Lie type. □

Lemma 4.13. *Let $L \in \mathcal{C}_2$ with $L/Z(L) \in \mathcal{R}$ and $K \in \mathcal{C}_2$ with $K \to L$, then $K/Z(K) \in \mathcal{R}$.*

Proof. Let $K = K_1, \dots, K_r = L$ be a chain which belongs to $K \to L$. We prove the lemma by induction on r. Hence it is enough to show that $K_{r-1}/Z(K_{r-1}) \in \mathcal{R}$. If $L \in \mathcal{L}^*_{K_{r-1}}$ this follows with Lemma 4.12. So let $L \in \mathcal{K}_{K_{r-1}}$. Then $K_{r-1} \cong L/N$, where $1 \neq N \leq Z(L)$. As L is perfect we have that $K_{r-1}/Z(K_{r-1}) \cong L/Z(L) \in \mathcal{R}$ again. □

Lemma 4.14. *Let $L \cong M(23)$. If $L \in \bar{\mathcal{L}}_K^*$, then $K = L$. Furthermore in this case L is not a component in the centralizer of a 2–central involution of G.*

Proof. Suppose false. By Proposition 4.5 we have that L is standard. Let U be a Sylow 2–subgroup of $C_G(L)$. By Lemma 3.4 we have that U is abelian.

Assume first that $m_2(U) = 1$. Let $u \in U$ be an involution. As for all involutions x we have $O(C_G(x)) = 1$ also $O(C_G(u)) = 1$. As U is cyclic we have that $C_G(L)$ has a normal 2-complement and so $U = C_{C_G(L)}(u)$. By Lemma 2.11 we have that L possesses no nontrivial central extensions and no outer automorphism. Hence we get that $C_G(u) = U \times L$. Further $N_G(L) = O(N_G(L))C_G(u)$. As U is cyclic we have that $u \notin C_G(u)'$ but $x \in C_G(x)'$ for all involutions $x \in L$ by Lemma 2.11, we get that

$$u^G \cap L = \emptyset. \tag{1}$$

Let T be a Sylow 2–subgroup of $N_G(L)$ containing u. By Lemma 2.11 we have that all involutions in L are 2–central and $Z(T) \cap L$ is of order 4. So we get that $Z(T) = \langle U, z, t\rangle$, where $\langle z, t\rangle = Z(T) \cap L$ is elementary abelian. Now as by (1) $u^G \cap (T \cap L) = \emptyset$, we get with Lemma 2.3 that u is not 2–central. In particular $N_G(Z(T)) \not\leq C_G(u)$ and so $u^{N_G(Z(T))} \neq \{u\}$. As $\Phi(Z(T)) \leq \langle u\rangle$ we see $U = \langle u\rangle$.

By Lemma 2.11 $J(T) = F$ is elementary abelian of order 2^{12}. Furthermore with $E = F \cap L$ we have $N_L(F)/E \cong M_{23}$ and $N_L(E)$ induces on $E^\sharp$ orbits of length 23, 253 and 1771. We have $Z(T) \leq F$ and as F is characteristic in T and T is not a Sylow 2-subgroup of G, we get $|N_G(F)|_2 > |C_G(u)|_2$, hence $|u^{N_G(F)}|$ is even and $u^G \cap E = \emptyset$ by (1). In particular for $|u^{N_G(F)}|$ we get the possibilities $1+23$, $1+253$, $1+1771$ and $1+23+253+1771$. As $|u^{N_G(F)}|$ has to divide $|GL_{12}(2)|$ and 443 does not divide $|GL_{12}(2)|$ we see that $|u^{N_G(F)}| \neq 1+1771$.

Assume first that $|u^{N_G(F)}| = 254$. Then

$$|N_G(F)/C_G(F)| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 127.$$

For $\omega \in N_L(F)$, $o(\omega) = 23$, we have that $C_F(\omega) = \langle u \rangle$. Now we choose $t \in F$ such that $|t^{N_L(F)}| = 23$. Then also 23 divides $|t^{N_G(F)}|$. This implies $|t^{N_G(F)}| = 23$, 46, $23 \cdot 127$ or $46 \cdot 127$. But the last two numbers cannot be written as a sum of some numbers from 23, 23, 253, 1771 and 1771. So we have $|t^{N_G(F)}| = 23$ or 46. In both cases an element ν of order 127 has to centralize all conjugates of t. As $\langle t^{N_G(F)} \rangle \geq E$, we get the contradiction $[\nu, F] = 1$.

Assume next that $|u^{N_G(F)}| = 2^{11}$. As $u^{N_G(F)} \cap E = \emptyset$ and $|F \setminus E| = 2^{11}$, this gives $uE = u^{N_G(F)}$ and so all elements in $F \setminus E$ are conjugate. This implies $E \trianglelefteq N_G(F)$. Further

$$|N_G(F)/F| = 2^{11}|N_{C_G(u)}(F)/F| = 2^{18} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

According to Lemma 2.11 there is $t \in E$ such that $C_L(t) \cong 2M(22)$. Now the possibilities for $|t^{N_G(F)}|$ are 23 or $23+253 = 276$, $23+1771 = 1794$ or $23+253+1771 = 2^{11}-1$. As $|t^{N_G(F)}|$ has to divide $|N_G(F) : C_{N_L(F)}(t)|$, we see that $|t^{N_G(F)}|$ has to divide $2^{18} \cdot 23$ and so $|t^{N_G(F)}| = 23$. Hence 2^{18} divides $|C_{N_G(F)}(t)/F|$.

Suppose that $E(C_G(t)) = 1$. As $F(C_{C_G(u)}(t)) = \langle t, u \rangle$, we get that $\langle u, t \rangle$ contains $C_{F(C_G(t))}(u)$ and so by [KuSte, 5.3.10] $F(C_G(t))$ is contained in a dihedral or semidihedral group. But $C_L(t)$ contains $2M(22)$ and so a group $M(22)$ has to act faithfully on $F(C_G(t)) = F^*(C_G(t))$, a contradiction. This shows that $E(C_G(t)) \neq 1$.

Let M be some component of $C_G(t)$. Suppose that $[M, u] \leq M$. Assume first $[u, M] = 1$, then $M \cong 2M(22)$. Now again $C_{C_G(M)}(u) = \langle t, u \rangle$. So $C_G(M)$ has dihedral or semidihedral Sylow 2–subgroups by [KuSte, 5.3.10]. This shows that $|N_{C_G(M)}(\langle t, u \rangle) : C_{C_G(M)}(\langle t, u \rangle)| \leq 2$. From

$F \cap M = F \cap L = E$ and $F = E\langle u\rangle$, we conclude that $N_{C_G(t)}(F) = N_M(E)N_{C_G(t)}(\langle u, t\rangle)$. As $N_M(F)/F \cong M_{22}$ by Lemma 2.11, we now see that $|N_{C_G(t)}(F)/F|$ divides $2 \cdot |M_{22}|$, which contradicts that 2^{18} divides $|C_{N_G(F)}(t)/F|$.

So $[M, u] \neq 1$. Assume next that $C_M(u)\langle t\rangle/\langle t\rangle$ has a component $M(22)$. As $C_G(u) \cap C_G(t)$ does not contain a subgroup $M(22)$, we see that $C_L(t) \leq M$ and so $t \in Z(M)$. Application of Lemma 2.56 shows $M/Z(M) \in \mathcal{M}$. By Lemma 2.58 we see that $M/Z(M) \cong {}^2E_6(2)$, F_2 or $M(22)$. But by Lemma 2.17 only $M \cong 2M(22)$ is left, which shows that $[M, u] = 1$, a contradiction. So we have $C_M(u) \leq \langle t, u\rangle$. Then first of all $t \in M$ and again by [KuSte, 5.3.10] $M\langle u\rangle$ has dihedral or semidihedral Sylow 2–subgroups. Furthermore as $t \in M$, we have by Lemma 2.56 $M/Z(M) \in \mathcal{M}$. Furthermore we now have that $M/Z(M)$ has a dihedral Sylow 2-subgroup. But there are no groups with a dihedral or semidihedral Sylow 2–subgroup in $\mathcal{M}$.

So we have that $M^u \neq M$. Then u centralizes a diagonal, which again shows that $M \cong 2M(22)$, $MM^u = M \times M^u$ and $C_{MM^u}(u) \leq L$. In particular $E \leq C_{MM^u}(u)$. Let E_1 be the projection of E onto M and $E_2 = E_1^u$. Then $[E_1, E_2] = 1$, $[E, E_1] = 1$ and $[u, E_1] \leq E$. Suppose $t \in E_1 \cap E_1^u$. Then $|[E_1, u]| = 2^{10}$ and is invariant under $N_{C_L(t)}(E)$. But by Lemma 2.11 there is no 10-dimensional submodule in E. Hence we have that $E_1E_1^u = E_1 \times E_1^u$. We have $E_1 \leq N_G(F)$ and $|C_{N_G(F)}(E)/F| \geq |E_1| = 2^{11}$. Furthermore as E is normal in $N_G(F)$, $N_G(F)/C_{N_G(F)}(E)$ involves M_{23} and $|N_G(F)/F| = 2^{11}|M_{23}|$, we see that $E_1F = C_{N_G(F)}(E)$ Now we conclude that there is $E_1 \leq N_G(F)$, E_1 elementary abelian of order 2^{11} such that $u^{E_1} = uE$. This gives that

$$N_G(F)/F \cong 2^{11}M_{23}.$$

We set $F_1 = O_2(N_G(F))$. Then we see

$$[F_1, E] = 1.$$

We have that $F_1 = (E_1 \times E_1^u)\langle u\rangle$. So $N_G(F_1) = N_G(E_1 \times E_1^u)$. As $u^{F_1} = uE$, we see that $N_G(F_1) = N_G(F)$. Hence

$$(*) \qquad F_1 \text{ is a Syow 2-subgroup of } C_G(E).$$

Let now $K \to L$ and $K = K_1, \ldots, K_r = L$ be the corresponding chain. By Lemma 4.6(b) we have $K_{r-1} \leq^* L$. So let $L_1 \sqsubseteq L$ with $L_1 \neq L$ and $L_1 \in \mathcal{C}_2$. By definition of $\sqsubseteq$ there is some involution w such that L_1 is a component of $C_G(w)$. By Lemma 2.11 we have $L_1 = C_L(t)$ or

$L_1 \cong 2^2U_6(2)$. In particular

$$(**) \qquad |L_1|_2 \leq 2^{18}.$$

In both cases $E \leq L_1$ and so $C_G(L_1) \leq C_G(E)$. So by $(*)$ we may assume that $w \in F_1$. As $w \not\sim u$, since $C_G(u)$ does not contain such components L_1, we see that $w \in E_1 \times E_1^u$. But then $E_1 \leq C_G(w)$ and as L_1 is a component and as $[E, E_1] = 1$, we see that $[E_1, L_1] \leq L_1$. As $C_{L_1}(E) = E$, we see that $E_1E = C_{L_1E_1}(E)$. In particular $N_{L_1}(E)$ acts on E_1E/E. As $N_{L_1}(E)$ acts on E_1E/E the same way as on E, we see with Lemma 2.11 that $|[N_{L_1}(E), E_1E/E]| \geq 2^9$. As $[N_{L_1}(E), EE_1] \leq L_1$ this shows that $|EE_1 \cap L_1| \geq 2^{19}$, contradicting $(**)$.

So we have shown that

$$(2) \qquad |u^{N_G(F)}| = 24.$$

As $N_L(F)$ acts 4-fold transitively on $u^{N_G(F)} \setminus \{u\}$, we get that $N_G(F)$ acts 5-fold transitively on u^G, so $N_G(F)/F \cong M_{24}$ by [Asch5, (18.10)]. Let S be a Sylow 2–subgroup of $N_G(F)$ which contains T. As M_{24} does not posses a failure of factorization module by [GoLyS1, Theorem 32.2] we get with [GoLyS1, Theorem 8.6] that $J(S) = F$ again, and so S is a Sylow 2–subgroup of G. Furthermore by Lemma 2.12 the action on F is well defined and we have orbits of length 24, 276, 1771 and 2024. So let now $s \in Z(S)$ be an involution, then $C_{N_G(F)}(s)/F \cong 2^63\Sigma_6$ by Lemma 2.12 again. As by Lemma 2.13 we have that $S = N_{N_G(F)}(S)$ and we just have one orbit of odd length, we see $|Z(S)| = 2$. As G is not of even characteristic, we have that $C_G(s)$ possesses a component M.

Suppose first that $N_{C_G(s)}(M) \cap N_G(F) \leq O_{2,3}(N_{C_G(s)}(F))$. Then the Σ_6 in $C_{N_G(F)}(s)/F$ acts nontrivially on $|M^{C_G(s)}|$ and so $|M^{C_G(s)}| \geq 6$. As $|S| = 2^{22}$ and $|S : S \cap N_{C_G(s)}(M)| \geq 16$, we see $|\langle M^{C_G(s)}, s\rangle/\langle s\rangle|_2 \leq 2^{17}$. As $|\langle M^{C_G(s)}, s\rangle/\langle s\rangle|_2 \geq |M\langle s\rangle/\langle s\rangle|_2^6$, we get that $|M\langle s\rangle/\langle s\rangle|_2 = 4$. As a Sylow 2-subgroup of M cannot be a quaternion group of order 8 by the definition of $\mathcal{C}_2$ then $s \notin Z(M)$. So S possesses an elementary abelian subgroup of order 2^{13}, which contradicts $F = J(S)$.

So we have that $O^2(C_{N_G(F)}(s)) \leq N_G(M)$. As the outer automorphism group of M is solvable, we get that $O^2(C_{N_G(F)}(s))$ induces inner automorphisms. Suppose $O^2(C_{N_G(F)}(s)) \not\leq M$. By Lemma 2.12 we have that $|C_{N_G(F)}(s) : O^2(C_{N_G(F)}(s))| \leq 4$. Set $R = MC_{O^2(C_{N_G(F)}(s))}(M)$. If $O^2(C_{N_G(F)}(s)) \not\leq R$, then $O^2(C_{N_G(F)}(s))/O^2(C_{N_G(F)}(s)) \cap R$ involves a group A_6. We have that

$$2^{22} \geq |O^2(C_{N_G(F)}(s))|_2 |O^2(C_{N_G(F)}(s))/C_{O^2(C_{N_G(F)}(s))}(M)|_2.$$

But the first factor is at least 2^{20} by Lemma 2.12 while the second factor is of order at least 8, as just seen. This contradiction shows $[O^2(C_{N_G(F)}(s)), M] = 1$. Now as $|S| = 2^{22}$ and $|O^2(C_{N_G(F)}(s))|_2 = 2^{20}$, we get that $|M\langle s\rangle/\langle s\rangle|_2 = 4$. This gives that $|M|_2 = 4$ as $M \in \mathcal{C}_2$ and so $s \notin M$. But as S normalizes M, as $S \leq MO^2(C_{N_G(F)}(s))$ and so $Z(S) \cap M \neq 1$, which contradicts $\langle s\rangle = Z(S) \not\leq M$. So we have that $O^2(C_{N_G(F)}(s)) \leq M$ and then $s \in M$. This shows $M/Z(M) \in \mathcal{M}$ by Lemma 2.56. As $|C_{N_G(F)}(s)/O^2(C_{N_G(F)}(s))| \leq 4$, we get that $2^{19} \leq |M/\langle s\rangle|_2 \leq 2^{21}$. By Lemma 2.58 we see $M/Z(M) \cong Co_1$. But in Co_1 by [GoLyS3, Table 5.3l] there is a unique elementary abelian subgroup $\tilde{E}$ in $S \cap M/\langle s\rangle$ of order 2^{11} which is normalized by M_{24}. As $F = J(S)$, we get that $\tilde{E} = F/\langle s\rangle$. But this contradicts the structure of $C_{N_G(F)}(s)$. This final contradiction shows that the assumption $m_2(U) = 1$ is false.

So we now assume that $m_2(U) > 1$ and then by Lemma 3.4 U is elementary abelian. By [Asch1, Theorem 2] we get some $g \in G$ such that $U^g \leq N_G(L)$ and $U^g \cap C_G(L) = 1$. Let T_1 be a Sylow 2–subgroup of L, then $U \times T_1 = T$ is a Sylow 2–subgroup of $N_G(L)$. By Lemma 2.11 all involutions in T are 2-central in $N_G(L)$. So we may assume that $U^g \cap Z(T) \neq 1$. Hence there is some $u \in U^\sharp$ such that $u \neq u^g \in Z(T)$. Then T is a Sylow 2–subgroup of $C_G(u^g)$ as well and so we may assume that $g \in N_G(T)$. In particular $U^g \leq Z(T)$.

Recall $Z(T_1) = \langle z, t\rangle$ by Lemma 2.11. Then as $U^g \cap U = 1$, we get that $|U| = 4$. Again by Lemma 2.11 there is some element ρ of order three in L such that $[\rho, z] = 1$ but $[\rho, t] \neq 1$. Hence $C_{Z(T)}(\rho) = \langle U, z\rangle$. Now for some $u \in U^\sharp$ we have $[u, g] \in C_{Z(T)}(\rho)$, as $|[U, g]| = 4$. Hence $(u^g u)^{\rho^{-1}} = u^g u$. This gives $u^g = u^{g\rho^{-1}}$ and then $u = u^{g\rho^{-1}g^{-1}}$. We obtain that $u^{g(g^{-1})^\rho} = u$, and so $g(g^{-1})^\rho \in C_G(u)$. We have that $C_{C_G(L)}(u)/\langle u\rangle$ has a Sylow 2–subgroup of order 2 and so has a normal 2–complement. As $O(C_G(u)) = 1$, we get that $C_G(u) = U \times L$. This now shows that $[U, g(g^{-1})^\rho] = 1$. Choose $v \in U^\sharp$ arbitrarily. Then $v^{g\rho^{-1}g^{-1}\rho} = v$. So $v^{g\rho^{-1}} = v^g$, i.e. $[v^g, \rho] = 1$. But then $U^g \leq C_{Z(T)}(\rho)$ and then $U^g \leq \langle U, z\rangle$, which gives $U \cap U^g \neq 1$, a contradiction. □

Lemma 4.15. *Let G be a group of even type, which is not of even characteristic. If G has a standard subgroup $L \cong 2M(22)$, then $G \cong M(23)$.*

Proof. Application of [DaSo, (4.4)] shows that either $C_G(Z(L)) = L$ and then by [Asch4, Theorem 32.1] we get $G \cong M(23)$, or $N_G(L) \cong 2\mathrm{Aut}(M(22))$. In the latter [DaSo, (4.6)] shows that centralizers of 2–central involutions do not have components, so G would be of even characteristic, a contradiction. □

Lemma 4.16. *$2M(22)$ is not in $\bar{\mathcal{L}}^*_K$ for any $K \in \mathcal{C}_2$. The same is true for $M(23)$ if we add that K is a component in $C_G(z)$ for some 2–central involution z.*

Proof. The second assertion is Lemma 4.14. Let $2M(22)$ in $\bar{\mathcal{L}}^*_K$. Then by Proposition 4.5 $2M(22)$ is a standard subgroup. By Lemma 4.15 we get that G is not a counterexample as it now possess a standard subgroup with cyclic centralizer, which belongs to a 2-central involution centralizer. □

4.1. **Nonsimple components.** In this subsection we will show that there are standard subgroups for 2-central involutions provided there are 2-central involutions with nonsimple components in their centralizer.

Hypothesis 4.17. There is $z \in \Omega_1(Z(S))^\sharp$ with $z \in Z(E(C_G(z)))$. By A_z we denote some component of $C_G(z)$ with $Z(A_z) \neq 1$. Further we assume that if $z \in Z(A_z)$ we have that A_z is not standard.

Lemma 4.18. *Assume Hypothesis 4.17 Then $\bar{\mathcal{L}}^*_{A_z} \cap \mathcal{M}_2 = \emptyset$.*

Proof. Suppose false. Pick $L \in \bar{\mathcal{L}}^*_{A_z} \cap \mathcal{M}_2$. Then $L/Z(L) \in \mathcal{R}$. By Lemma 4.13 we have $A_z/Z(A_z) \in \mathcal{R}$. As $Z(A_z) \neq 1$ we get $A_z/Z(A_z) \ncong M(23)$, $M(24)'$ or F_1. By Lemma 4.16 $L \ncong 2M(22)$ or $M(23)$. We have $L/Z(L) \in \mathcal{R} \cap \mathcal{M}_2$, which gives

$$L \cong M(22), M(24)', F_2, 2F_2 \text{ or } F_1. \tag{1}$$

By Proposition 4.5 L is standard and so by Lemma 3.4 $C_G(L)$ has abelian Sylow 2–subgroups U, which in fact are elementary abelian if $m_2(U) > 1$.

Assume first that there is $u \in U$, $u \sim z$ in G. Denote by A_u the conjugate of A_z in $C_G(u)$. Then $Z(A_u) \neq 1$. But as $C_G(u) \leq N_G(L)$ we get $A_u = L$ as $\mathcal{R}$ does not contain groups with abelian Sylow 2–subgroups. This shows $L \cong 2F_2$. Furthermore L and so also A_z are standard. Now A_z is normal in $C_G(z)$ and so $Z(A_z) \cap Z(S) \neq 1$. As A_z is standard we have a contradiction. So we have that

$$z^G \cap U = \emptyset.$$

Let now T be a Sylow 2–subgroup of $N_G(L)$ with $U \leq T$. By Lemma 2.67 we see that $|Z(T/U)| = 2$, as $L \not\cong M(23)$. As $Z(S)$ is conjugate to a subgroup of $Z(T)$ there is $v \in Z(T) \setminus U$ such that $v \sim z$ in G. Again denote by A_v the component of $C_G(v)$ which is conjugate to A_z.

By (1)

$$L \cong M(22), M(24)', F_2, 2F_2 \text{ or } F_1.$$

So by Lemma 2.10 we have in the corresponding order that

$$C_L(v) \cong 2 \cdot 2^{1+8}U_4(2) : 2, 2^{1+12}3U_4(3) : 2, 2^{1+22}Co_2,$$
$$2 \cdot 2^{1+22}Co_2 \text{ or } 2^{1+24}Co_1.$$

Assume there is $u \in \Omega_1(U)^\sharp$ with $[A_v, u] = 1$ or $A_v^u \neq A_v$. Then there is a component $\tilde{L}$ in $C_{A_v^u A_v}(u)$ which is a component of $C_{C_G(u)}(v)$ and so it is in $C_G(L)$. As L is standard, $C_G(L)$ has abelian Sylow 2-subgroups. But then also $\tilde{L}/Z(\tilde{L}) \cong A_v/Z(A_v)$ now has abelian Sylow 2–subgroups. As $A_v/Z(A_v) \in \mathcal{R}$ we have a contradiction.

Hence we have $[A_v, u] = A_v$ for all $u \in \Omega_1(U)^\sharp$. Set

$$X_v = O^2(C_L(v)) \text{ and } L_v = \langle A_v^{X_v} \rangle.$$

Then $C_{L_v}(u) \cap X_v \trianglelefteq X_v$. If $X_v \leq C_{L_v}(u)$, then $L_v = A_v$ and then even $X_v \leq A_v$. Now $|A_z|_2 = |A_v|_2 \geq |L|_2/2$. But by Lemma 4.6(a) $|L|_2 \geq |A_z|_2$. Recall that by (1) $L \cong M(22)$, $M(24)'$, F_2, $2F_2$ or F_1. So by Lemma 2.58 we see $|A_z|_2 = 2^{16}$, 2^{17}, 2^{20}, 2^{21}, 2^{40}, 2^{41}, 2^{42}, 2^{45} or 2^{46}. As $|A_z|$ has to have a Sylow 2–subgroup of order at least 2^{16} and $A_z/Z(A_z) \in \mathcal{R}$ and $Z(A_z)$ is not trivial we get with Lemma 2.59 that $A_z/Z(A_z) \cong U_6(2)$, $F_4(2)$, ${}^2E_6(2)$, $M(22)$ or F_2. Now the orders of the Sylow 2–subgroups of $A_z/Z(A_z)$ are 2^{15}, 2^{24}, 2^{36}, 2^{17} or 2^{41}. As $|Z(A_z)|$ is even, we get $|A_z|_2 = 2^{16}$, 2^{25}, 2^{37}, 2^{18} or 2^{42}. If we compare with the orders above, we see that just $A_z \cong 2U_6(2)$ or $2F_2$ is possible. In the first case $L \cong M(22)$, while in the second $L = A_z$. Hence in the second case we get $z \in L$, a contradiction. So we have the first case. Now $C_L(v) \neq X_v$. As $Z(C_L(v))$ is of order 2, we have that $Z(A_v)$ is not normalized by $C_L(v)$. But X_v is normalized by $C_L(v)$ and so A_v is normalized by $C_L(v)$, which shows that $Z(A_v)$ is normalized by $C_L(v)$ too, a contradiction.

Hence $C_{L_v}(u) \cap X_v \leq O_{2,3}(X_v)$. Suppose that $C_{L_v}(u) \cap X_v \not\leq O_2(X_v)$. Then by Lemma 2.10 $L \cong M(24)'$ and $O_{2,3}(X_v) \leq L_v$. We have that $C_{A_v}(u) \cap X_v$ is a $\{2,3\}$–group and $C_{L_v}(u) = \langle C_{A_v}(u)^{X_v} \rangle$. As $O_{2,3}(X_v)/O_2(X_v)$ is cyclic we get again $A_v = L_v$ and so $O_{2,3}(X_v) \leq A_v$.

But then X_v induces inner automorphisms on A_v and so $|A_v|_2 \geq 2^{20}$. As before this shows that $A_v/Z(A_v) \cong F_4(2)$, ${}^2E_6(2)$ or F_2. But $|A_v|_2 \leq 2^{21}$ by Lemma 4.6(a), a contradiction.

So we have $C_{L_v}(u) \cap X_v \leq O_2(X_v)$. In particular $C_{L_v}(u) \leq C_G(L)O_2(X_v)$ for all $u \in \Omega_1(U)^\sharp$. We now choose $u \in \Omega_1(U)^\sharp$ with $|C_{L_v}(u)|_2$ maximal. Let T_1 be a Sylow 2–subgroup of $C_{L_v}(u)$. As T_1U is normalized by X_v we see $T_1U \geq O_2(X_v)$ or $T_1U = UZ(O_2(X_v))$. In particular $Z(T_1U) \leq UZ(O_2(X_v))$. If $|\Omega_1(U)| \geq 8$, we get $N_G(T_1U) \leq N_G(U)$, as U is tightly embedded and $|Z(O_2(X_v))| \leq 4$. Now choose $T_2 \leq L_v$ with $|T_2 : T_1| = 2$ and $[U, T_2] \leq T_1U$. Then $T_2 \leq N_G(U)$ and so there is some $1 \neq \tilde{u} \in U^\sharp$ with $[T_2, \tilde{u}] = 1$, contradicting the maximality of $|C_{L_v}(u)|_2$. So we have that T_1 is a Sylow 2–subgroup of L_v and then $T_1 \leq UO_2(X_v)$ has class at most two, a contradiction to $A_v/Z(A_v) \in \mathcal{R}$ and Lemma 2.67.

So we $|\Omega_1(U)| \leq 4$. Then $C_{C_G(L)}(u)/\langle u \rangle$ has cyclic Sylow 2–subgroups and so a normal 2–complement, which shows $C_{C_G(L)}(u) = U$. In particular $C_{L_v}(u)$ is a 2–group, which contradicts Lemma 2.66 . $\square$

Lemma 4.19. *Assume Hypothesis* 4.17. *Then* $\mathcal{L}^*_{A_z} = \bar{\mathcal{L}}^*_{A_z}$.

Proof. By Lemma 2.56 we have $A_z/Z(A_z) \in \mathcal{M}$. Choose $L \in \mathcal{L}^*_{A_z}$ with $L \neq A_z$. Then by Lemma 4.9 $L \in \mathcal{M}_1$. If $L \notin \bar{\mathcal{L}}^*_L$, then by Lemma 4.10 $\bar{\mathcal{L}}^*_L \subseteq \mathcal{M}_2$, which contradicts Lemma 4.18. So $L \in \bar{\mathcal{L}}^*_L$, i.e. $\mathcal{L}^*_{A_z} = \bar{\mathcal{L}}^*_{A_z}$.

So we are left with $\mathcal{L}^*_{A_z} = \{A_z\}$. Now choose $L \in \bar{\mathcal{L}}^*_{A_z}$. Let $A_z, K, \cdots, L$ be a chain for $A_z \longrightarrow L$. Then we have that $K \in \mathcal{K}_{A_z}$. Hence $K/N \cong A_z$ for some $1 \neq N \leq Z(K)$. In particular $|Z(K)| \geq 4$, as $Z(A_z) \neq 1$. By definition $K \notin \bar{\mathcal{L}}^*_K$. Hence there is L_1 with $K \sqsubseteq L_1$. By Lemma 2.60 (b) and (c) we have that $L_1 \in \mathcal{M}_2$. Further $L \in \bar{\mathcal{L}}^*_{L_1}$. But then $L \in \mathcal{M}_2$ by Lemma 4.10, contradicting Lemma 4.18. $\square$

Lemma 4.20. *Assume Hypothesis* 4.17. *If* $L \in \bar{\mathcal{L}}^*_{A_z}$, *then* $Z(L) \cap Z(A_z) \neq 1$.

Proof. By Lemma 4.19 we have $L \in \mathcal{L}^*_{A_z}$. By Lemma 2.56 we have $A_z \in \mathcal{M}$. Then by Lemma 4.9 we get $L \in \mathcal{M}_1$. By Lemma 4.18 we have $L \notin \mathcal{M}_2$. Then $Z(L) \neq 1$. By Lemma 4.19 we have $A_z \leq L$. If $Z(L) \cap Z(A_z) = 1$, we have a quasisimple component in the centralizer of an involution in $L/Z(L)$, which is not simple. By Lemma 2.60(b) we then get $L/Z(L) \in \mathcal{M}_2$. This also implies $L \in \mathcal{M}_2$, a contradiction. $\square$

Lemma 4.21. *Let* $z \in \Omega_1(Z(S))^\sharp$ *with* $Z(E(C_G(z))) \neq 1$, *then there is* $t \in \Omega_1(Z(S))^\sharp$ *with* $t \in Z(E(C_G(t)))$.

Proof. Choose $1 \neq t \in \Omega_1(Z(S)) \cap Z(E(C_G(z)))$. As

$$\begin{aligned}[E(C_G(z)), O_2(C_G(t))] &\leq O_2(C_G(t)) \cap E(C_G(z)) \\ &\leq O_2(E(C_G(z))) \leq Z(E(C_G(z))),\end{aligned}$$

we get with the 3–subgroup lemma that $[E(C_G(z)), O_2(C_G(t))] = 1$. So we have that $E(C_G(t)) \neq 1$.

Let L be some component of $E(C_G(z))$. Assume first $L \cap E(C_G(t)) \leq Z(L)$. Choose $\rho \in L$, $o(\rho) = p > 2$, p prime. Assume there is a component K of $C_G(t)$ with $K^\rho \neq K$. Then $K^{\langle\rho\rangle} = K_1K_2 \cdots K_p$. Now choose $1 \neq x_1 \in S \cap K_1$, $x_1 \notin Z(K_1)$. Then $\langle x_1^{\langle\rho\rangle}\rangle$ is a 2–group and $\langle x_1^{\langle\rho\rangle}\rangle \neq \langle x_1\rangle$. As $x_1 \in S$ we have $x_1 \in C_G(z)$. So $\rho^{-1}\rho^{x_1} \in LL^{x_1}$. As $[x_1, \rho]$ is a 2–element we have that $L^{x_1} = L$ and so $[L, x_1] \leq L \cap E(C_G(t)) \leq Z(L)$, but then $[x_1, \rho] = 1$, a contradiction.

This shows that L normalizes any component of $C_G(t)$. As L centralizes $O_2(C_G(t))$, we see that

$$LF^*(C_G(t)) = F^*(C_G(t))C_{LF^*(C_G(t))}(F^*(C_G(t))) = F^*(C_G(t)).$$

So we have $L \leq E(C_G(t))$. This gives $E(C_G(z)) \leq E(C_G(t))$ and so $t \in Z(E(C_G(t)))$. □

Proposition 4.22. *Suppose that there is some $z \in \Omega_1(Z(S))^\sharp$ such that $Z(E(C_G(z))) \neq 1$. Then we may choose z and some component A_z of $C_G(z)$ with $z \in Z(A_z)$ and A_z is standard.*

Proof. By Lemma 4.21 we can choose z such that $z \in Z(E(C_G(z)))$. Denote by A_z some component of $C_G(z)$ with $Z(A_z) \neq 1$. If A_z is standard, we have that $\Omega_1(Z(S)) \cap Z(A_z) \neq 1$ and so we may assume $z \in A_z$. Hence we just have to show that A_z is standard. So assume false. Then Hypothesis 4.17 is satisfied.

By Lemma 4.20 $Z(A_z) \cap Z(L) \neq 1$ for $L \in \bar{\mathcal{L}}^*_{A_z}$. So let $1 \neq t \in Z(A_z) \cap Z(L)$ be an involution. By Proposition 4.5 L is standard. Then we may assume that t is not 2-central in G. Now $N_G(L) \geq C_G(t)$ and so $E(C_G(z)) \leq C_G(t) \leq N_G(L)$. Assume first that A_z is normalized by $C_G(z)$. Then $Z(A_z)$ contains a 2–central involution and so we may assume that $z \in Z(A_z)$. As $A_z \leq L$ and $z \notin Z(L)$, we have a component in $C_{L/Z(L)}(zZ(L))$. By Lemma 4.9 we have $L \in \mathcal{M}_1$. But by Lemma 2.60(b) now $L \in \mathcal{M}_2$, contradicting Lemma 4.18.

So we have A_z is not normal in $C_G(z)$. We have $\langle A_z^{C_G(z)}\rangle \leq N_G(L)$ and $C_G(L) \cap \langle A_z^{C_G(z)}\rangle$ is normal in $\langle A_z^{C_G(z)}\rangle$. But by Lemma 2.57 A_z

has nonabelian Sylow 2–subgroups. So we get that $C_G(L) \cap \langle A_z^{C_G(z)} \rangle \leq Z(\langle A_z^{C_G(z)} \rangle)$. In particular $C_{L/Z(L)}(zZ(L))$ has more than one component, a contradiction to Lemma 2.64. □

4.2. **Simple components.** Now we show that there is always some 2-central involution whose centralizer contains a standard subgroup. By Proposition 4.22 we may work under the following assumption:

Hypothesis 4.23. Let S be a Sylow 2-subgroup of G. Assume that for all $z \in \Omega_1(Z(S))^\sharp$ we have $Z(E(C_G(z))) = 1$. Furthermore if A_z is a component of $C_G(z)$, then A_z is not a standard subgroup.

For the remainder, we fix $z \in \Omega_1(Z(S))^\sharp$ with $E(C_G(z)) \neq 1$, and we fix a choice of component A_z of $C_G(z)$. When $v \in z^G$, we shall denote by A_v a fixed G-conjugate of A_z.

Lemma 4.24. *Assume Hypothesis* 4.23*. Then* $|\Omega_1(Z(S))| \geq 4$.

Proof. We have that

$$\Omega_1(Z(S)) \cap O_2(C_G(z)) \neq 1 \text{ and } \Omega_1(Z(S)) \cap E(C_G(z)) \neq 1. \qquad \square$$

Lemma 4.25. *Assume Hypothesis* 4.23*. If* $L \in \bar{\mathcal{L}}^*_{A_z}$*, then* $L/Z(L) \in \mathrm{Chev}(2)$ *or* $L \cong L_4(3)$*. Furthermore* $L \ncong Sp_4(2)' \cong A_6$.

Proof. By Proposition 4.5 L is standard. So by Hypothesis 4.23 $C_G(L)$ does not contain 2–central involutions. Let U be a Sylow 2–subgroup of $C_G(L)$ and T_1 be a Sylow 2–subgroup of $N_G(L)$ with $U \leq T_1$. Let furthermore $T_1 \leq T$, T a Sylow 2–subgroup of G. We have that $Z(T) \cap U = 1$. As $C_G(L)$ is tightly embedded, we see that $Z(T) \leq T_1$. So by Lemma 4.24 $|\Omega_1(Z(T_1)) : \Omega_1(Z(T_1)) \cap U| \geq 4$. Hence $|\Omega_1(Z(T_1/U))| \geq 4$. This gives by Lemma 2.34 that $L \in \mathrm{Chev}(2)$, $L_4(3)$, or $L_2(9)$. Recall that $L \cong M(23)$ is not possible by Lemma 4.14. As $L \neq A_z$, we see that $\mathrm{Aut}(L)$ contains an involution with nonsolvable centralizer. So we get $L \ncong L_2(9)$. □

Lemma 4.26. *Assume Hypothesis* 4.23*. If* $L \in \bar{\mathcal{L}}^*_{A_z}$*, then* $L \ncong L_4(3)$.

Proof. Suppose $L \cong L_4(3)$. As in the proof of Lemma 4.25 we have some Sylow 2–subgroup T of G which contains a Sylow 2–subgroup T_1 of $N_G(L)$ and so $Z(T) \leq T_1$. As $|\Omega_1(Z(T))| \geq 4$ and $Z(T) \cap C_G(L) = 1$ by Hypothesis 4.23, we have by Lemma 2.19 some outer automorphism x of L, where $x \in Z(T)$ such that $C_L(x) \cong PSp_4(3) : 2$. Set $U = T \cap C_G(L)$. Then $[U, x] = 1$. As U is abelian by Lemma 3.4, we get $|\Omega_1(Z(T_1))| = |U| \cdot 4$. As U does not contain 2–central involutions, we have that $T_1 \neq T$ and so there is $t \in N_T(T_1)$ with $U \cap U^t = 1$.

This shows $|U| \leq 4$ and as $Z(T_1/U)$ is elementary abelian again by Lemma 2.19, we have that U is elementary abelian. In particular:

$(*)$ For each $u \in U^\sharp$, we have that $C_G(u) = U \times L_4(3) : 2 \leq N_G(L)$.

As $L \in \bar{\mathcal{L}}^*_{A_z}$ and $A_z \neq L$, we have a chain $A_z = K_1, \ldots, K_r = L$ belonging to $A_z \to L$, where $r > 1$. We have $K_{r-1} \leq^* L$. Hence there is $L_1 \neq L$ with $L_1 \sqsubseteq L$. This now shows that $L_1 \cong PSp_4(3)$. Furthermore either $L_1 = K_{r-1} \cong PSp_4(3)$ or $K_{r-1} \cong L_2(9)$. In the second case we have $A_z = K_{r-1}$.

So we have

$$A_z \cong PSp_4(3) \text{ or } L_2(9).$$

Now we choose $v \in Z(T \cap N_G(L))$ with $v \sim z$ in G. Let $T_2 \leq T$ with $|T_2 : T_1| = 2$. We have that $T_1' \leq L$ by $(*)$. By Lemma 2.19 we have $|Z(T_1 \cap L)| = 2$ and so $Z(T_1 \cap L)$ is centralized by T_2. As $C_{T_1}(T_2) = Z(T)$ and $C_{T_1}(T_1) = U \times Z(T)$, we have that $Z(T) \cap L \neq 1$. So we see that $UL \cap Z(T) \leq L$. This gives that either $v \in L$ or v induces an outer automorphism on L.

Let A_v be some component of $C_G(v)$, which is conjugate to A_z. Suppose first $v \in L$. Then by Lemma 2.19 $C_L(v)$ contains $SL_2(3) * SL_2(3)$. Further $C_L(v)$ acts irreducibly on $O_2(C_L(v))/Z(O_2(C_L(v)))$. Set

$$L_v = \langle A_v^{C_L(v)} \rangle.$$

As $L_v \cap C_L(v)$ is normal in $C_L(v)$ and $v \notin L_v$ as $Z(E(C_G(v))) = 1$, we see that $C_L(v) \cap L_v = 1$.

Choose $u \in U^\sharp$ and set $X_u = C_{A_v^u A_v}(u)$. As $X_u \leq C_G(u) \cap C_G(v)$, we get that X_u is solvable. So $[A_v, u] = A_v$. As $C_L(v) \cap A_v = 1$, we have that $C_{A_v}(u) \leq U \times Z(T)$ and so is elementary abelian of order at most 16. But neither $L_2(9)$ nor $PSp_4(3)$ has such an automorphism u by Lemma 2.18.

So we have shown that v induces an outer automorphism on L and then $C_L(v) \cong PSp_4(3) : 2$. Again choose $u \in U^\sharp$.

Assume first that $[u, A_v] = 1$. Then $A_v \leq L$ and is normal in $C_L(v)$, so $A_v \cong PSp_4(3)$. Now $C_{N_G(L)}(A_v) = C_G(L) \times \langle v \rangle$.

Suppose $|U| \geq 4$. Then $U \times \langle v \rangle$ is a Sylow 2–subgroup of $C_{N_G(L)}(A_v)$. As U is a TI–group in $U \times \langle v \rangle$, we see that this is a Sylow 2–subgroup of $C_G(A_v)$. As $A_v \cong PSp_4(3)$ does not have elementary abelian Sylow 2–subgroups, we see that $A_v^T = A_v$. But then T normalizes $C_T(A_v) = U \times \langle v \rangle$ and so U contains 2–central involutions, a contradiction.

So we have that $|U| = 2$. Then $C_{C_G(A_v)}(u) = \langle u, v \rangle$ and then by [KuSte, 5.3.10] $C_G(A_v)$ has dihedral or semidihedral Sylow 2–subgroups. As $v \in Z(C_G(A_v) \cap C_G(v))$, we see that $C_G(A_v) \cap C_G(v)$ cannot have components by Hypothesis 4.23 and so is solvable. Hence as $O(C_G(v)) = 1$, we have with Lemma 2.7 that $C_G(v) \cap C_G(A_v)$ is dihedral, semidihedral or contains a normal subgroup $SL_2(3)$. The latter is not possible by Lemma 3.2.

As A_v does not have dihedral or semidihedral Sylow 2–subgroups we get that $A_v \trianglelefteq C_G(v)$. Let now $w \in A_v$ be a 2–central involution. Then $\langle w \rangle = O_2(C_{A_v}(w))'$ by Lemma 2.19. As $w \notin Z(E(C_G(w)))$ we get $O_2(C_{A_v}(w)) \cap E(C_G(w)) = 1$.

Suppose $E(C_G(w)) \neq 1$ and choose some component A_w of $C_G(w)$. We have that T has a subgroup of index two, which is a direct product of a dihedral or semidihedral group and a Sylow 2–subgroup of $PSp_4(3)$. In particular T has no abelian section of rank greater than 6. As $w \notin \langle A_w^{C_G(w)} \rangle$ we get that A_w has at most two conjugates in $C_G(w)$. Hence $O^2(C_G(w) \cap C_G(u))$ normalizes A_w. By Lemma 2.54 $O_2(C_{A_v}(w))$ induces inner automorphisms on A_w. If $C_{O_2(C_{A_v}(w))}(A_w) = \langle w \rangle$, there is an elementary abelian section of order 2^8 in $A_w O_2(C_{A_v}(w))$, a contradiction. So we get that $A_w \leq C_G(O_2(C_{A_v}(w)))$. The structure of $C_G(v)$ shows that a Sylow 2-subgroup of $C_G(O_2(C_{A_v}(w)))$ is the same as of $C_G(A_v)$ and so is dihedral or semidihedral. Hence a Sylow 2–subgroup of A_w is dihedral or semidihedral. In particular as $u \not\sim v$, we must have dihedral Sylow 2–subgroups. As $A_w \in \mathcal{C}_2$, we get $A_w \cong L_2(p)$, p prime, or $L_2(9)$. This now shows that even $O^2(C_{A_v}(w))$ centralizes A_w and then $C_G(w)$ has a normal subgroup $A_w \times SL_2(3) * SL_2(3)$ and so has a subnormal subgroup $SL_2(3)$, contradicting Lemma 3.2, recall that $O^2(C_{A_v}(w))$ cannot be contained in a component as $w \notin E(C_G(w))$.

So we have $E(C_G(w)) = 1$. If $C_G(w) \leq C_G(v)$ we again have a subnormal $SL_2(3)$, a contradiction to Lemma 3.2. So we have $C_G(w) \not\leq C_G(v)$.

If $\Omega_1(Z(T)) = \Omega_1(Z(O_2(C_G(w))))$ then $Z(T) = \langle w, v \rangle$ is normal in $C_G(w)$ and so $C_G(w) \leq C_G(v)$, a contradiction. So we deduce that $\Omega_1(Z(O_2(C_G(w)))) > \langle v, w \rangle$. We have that $O_2(C_G(w)) \leq C_T(A_v) \times$

$O_2(C_{A_v}(w))$. As we have $C_G(O_2(C_G(w))) \leq O_2(C_G(w))$ we get that $O_2(C_{A_v}(w)) \leq O_2(C_G(w))$. This now gives $O_2(C_G(w)) \cong V_4 \times Q_8 * Q_8$. As $C_G(w) \not\leq C_G(v)$, there is some 3–element ρ which acts nontrivially on $\Omega_1(Z(O_2(C_G(w)))$ and $[\rho, w] = 1$. Set

$$X_w = \langle O_{2,3}(C_{A_v}(w))/O_2(C_{A_v}(w)), \rho\rangle.$$

Then X_w acts on $O_2(C_G(w))/Z(O_2(C_G(w)))$. Hence we may choose ρ such that $[\rho, O_2(C_G(w))] = [\Omega_1(Z(O_2(C_G(w)))), \rho]$. This gives that Sylow 3–subgroups of X_w are elementary abelian and so $[\rho, O_2(C_{A_v}(w))] = 1$. This then implies $O_{2,3}(C_G(w)) \cong A_4 \times SL_2(3) * SL_2(3)$ and $C_G(w) = O_{2,3}(C_G(w))(C_G(v) \cap C_G(w))$. Then we have a subnormal $SL_2(3)$ contradicting Lemma 3.2.

Hence we have a final contradiction for $[u, A_v] = 1$. This implies $[u, A_v] \neq 1$ for all $u \in U^\sharp$. Set

$$L_v = \langle A_v^{C_L(v)}\rangle.$$

Suppose that $C_L(v) \cap L_v = 1$. As $C_{L_v}(u)$ is normalized by $C_L(v)$, we then have that $C_{L_v}(u) = U$. But this contradicts Lemma 2.18 and $A_v \cong A_z \cong L_2(9)$ or $PSp_4(3)$. So we have that $C_L(v) \cap C_{L_v}(u) \neq 1$ and so $C_L(v)' \leq L_v$. If $[u, A_v] \leq A_v$ we get $C_L(v)' \leq A_v$ and so $L_v = A_v = C_L(v)'$, as $A_v \cong L_2(9)$ or $PSp_4(3)$. But this contradicts $[u, A_v] \neq 1$.

So we have that $A_v^u \neq A_v$. Then $E(C_G(\langle v, u\rangle)) \geq C_{A_v^u A_v}(u) \cong A_v$ and so $A_v \cong PSp_4(3)$. As $[C_{A_v^u A_v}(u), U] = 1$, we see that U normalizes $A_v^u A_v$ and so $|U| = 2$. We have that $A_v A_v^u = E(C_G(v))$ as $C_{C_G(u)}(C_{A_v A_v^u}(u)) = \langle u\rangle$. Hence $|T : N_T(A_v)| = 2$.

As $O(N_G(L))C_G(u) = N_G(L)$, we see that $|C_G(u)|_2 = 2^9$. Let x be an involution in $C_{C_G(v)}(A_v A_v^u)A_v A_v^u$. Then $|C_G(x)|_2 \geq 2^{10}$ and so $x \not\sim u$ in G. Suppose that x is an involution which acts on A_v as an outer automorphism. Then we have $|C_{A_v}(x)|_2 \geq 2^4$ by Lemma 2.18 and so again $|C_{A_v A_v^u}(x)|_2 \geq 2^8$. As $[v, x] = 1$ and $v \notin A_v A_v^u$ by Hypothesis 4.23 we get $|C_G(x)|_2 \geq 2^{10}$. So u is not conjugate to any involution in $N_T(A_v)$. By Lemma 2.3 we get a subgroup of index 2 in G, a contradiction. $\square$

Lemma 4.27. *Assume Hypothesis* 4.23. *If $L \in \bar{\mathcal{L}}^*_{A_z}$, then $L \not\cong L_2(q)$, q even.*

Proof. Suppose false. Then as $L \in \bar{\mathcal{L}}^*_{A_z}$ and outer automorphisms of L just centralize $L_2(r)$, we see that $A_z \cong L_2(r)$ for some r. As $r \geq 4$, we also see that $q \geq 16$. Finally there is some involution $x \in N_G(L)$ such that $C_L(x) \cong L_2(t)$, $t^2 = q$. Hence

$$L \cong L_2(t^2), A_z \cong L_2(r), r \leq t.$$

Let U be a Sylow 2–subgroup of $C_G(L)$. Then by Hypothesis 4.23 we have that U does not contain 2–central involutions. Let T_1 be a Sylow 2–subgroup of L. As $|T_1| \geq 16$, we see that $U \times T_1$ is a characteristic abelian subgroup in $T \cap N_G(L)$, where T is a Sylow 2–subgroup of G containing a Sylow 2–subgroup of $N_G(L)$ which contains $U \times T_1$. This shows U is elementary abelian. Further

$$C_G(U \times T_1) = U \times T_1.$$

Now we may apply O'Nan's lemma (Lemma 2.6) to $U \times T_1$ with $\rho \in N_L(T_1)$, $o(\rho) = q-1$. This gives that either $U \sim T_1$ in G or all elements in $U \times T_1 \setminus T_1$ are conjugate to elements in U. Recall that as $|T_1| \geq 16$ the other possibilities of O'Nan's lemma do not appear.

Now let $v \in U \times T_1 \cap Z(T)$, $v \sim z$ in G and let A_v be the component in $C_G(v)$ conjugate to A_z. Let further $u \in U^\sharp$. Suppose that either $[A_v, u] = 1$, $A_v^u \neq A_v$ or $[A_v, u] = A_v$ and $C_{A_v}(u)$ is nonsolvable. Then $C_{A_v A_v^u}(u) \leq (C_G(u) \cap C_G(v))^\infty \leq C_G(L)$. Hence we may assume that there is some $1 \neq \tilde{u} \in U$, with $\tilde{u} \in C_{A_v A_v^u}(u)$. We have $E(C_G(v)) = A_v A_v^u C_{E(C_G(v))}(\tilde{u})$. Now also $C_{E(C_G(v))}(\tilde{u}) \leq C_G(L)$ as it is nonsolvable. But as T normalizes $E(C_G(v))$ and $U \leq T$, then $1 \neq Z(T) \cap E(C_G(v)) \leq C_{A_v A_v^u}(u) C_{E(C_G(v))}(\tilde{u})$, contradicting $Z(T) \cap C_G(L) = 1$ by Hypothesis 4.23.

So we have that

$$[A_v, u] = A_v \text{ and } C_{A_v}(u) \text{ is solvable.}$$

Suppose that some $u \in U^\sharp$ induces a field automorphism on A_v. Set $X_v = C_{A_v}(u)$. Then X_v is solvable and so $A_v \cong L_2(4)$, $X_v \cong \Sigma_3$. In particular $|U| \leq 4$ and then $U = C_{C_G(L)}(u)$. By Lemma 2.23 $O_3(X_v) \leq L$. Then $C_{UL}(O_3(X_v)) = U \times R$, R of odd order. But X_v is centralized by $\langle v, u \rangle \leq UL$, a contradiction. So we have that

$$[A_v, u] = A_v \text{ for all } u \in U^\sharp \text{ and any}$$
$$u \text{ induces an inner automorphism.}$$

Let X_v be a Sylow 2–subgroup of A_v, which is centralized by U. In particular $|X_v| \geq |U|$. As $X_v \leq N_G(L) \cap C(v)$, and T_1U is of index two in $\Omega_1(T \cap N_G(L))$, we see that T_1 centralizes a subgroup of index two in X_v. If $[T_1, X_v] \neq 1$, then as $q \geq 16$, we have that $|T_1 : C_{T_1}(X_v)| \geq 4$ and so

T_1 induces a fours group of outer automorphisms on A_v, contradicting $A_v \cong L_2(r)$. This shows that

$$[T_1, X_v] = 1 \text{ and then } X_v \leq U \times T_1.$$

Suppose first that $|U| = |T_1|$. Then as $|X_v| \geq |U|$, we have $A_v \cong L_2(r)$, $r \geq q$, a contradiction. So we have that $|U| < |T_1|$. In particular by O'Nan's lemma we now have that all elements in $U \times T_1 \setminus T_1$ are conjugate to elements in U, which gives that just the involutions in T_1 are 2–central.

Suppose that X_v does not contain 2–central involutions. Then $X_v \cap T_1 = 1$ and so $|X_v| = |U|$. As X_v is a Sylow 2–subgroup of A_v we see that T cannot normalize A_v. Hence there is some $y \in T$ with $A_v^y \neq A_v$. Now we could also have chosen A_v^y instead of A_v. Then also $X_v^y \leq T_1 \times U$ and so $X_v X_v^y \cap T_1 \neq 1$. Hence all involutions in $X_v X_v^y \setminus (X_v \cup X_v^y)$ are 2–central. But the set of 2–central involutions in $T_1 \times U$ is closed under multiplication, a contradiction.

So we have that X_v contains 2–central involutions and so $X_v \leq T_1$. Now we choose $\nu \in N_{A_v}(X_v)$, which acts transitively on $X_v^\sharp$. We have $[C_{C_G(v)}(A_v), \nu] = 1$. As $UT_1 = C_{UT_1}(A_v) \times X_v$ we have that ν normalizes UT_1 and $[UT_1, \nu] = X_v$. Now there is some $w \in UT_1$ with $[w, \nu] = 1$ and $w \sim u \in U^\sharp$ in G. Let $g \in G$ with $u^g = w$. Then $U \times T_1 \leq C_G(w)$ and so $U \times T_1$ is a Sylow 2–subgroup of $C_G(L^g)L^g$, as UT_1 was the only abelian subgroup of its order in a Sylow 2-subgroup of $N_G(L)$. But as $T_1^\sharp$ is the set of 2–central involutions in $U \times T_1$, we get that T_1 is a Sylow 2–subgroup of L^g. As ν normalizes L^g it now acts nontrivially on L^g. As $|X_v| < |T_1|$ and $[T_1, \nu] = X_v$, we see that ν induces a field automorphism on L^g. As $o(\nu)$ is odd, this implies $|[T_1, \nu]| > t$, where $q = t^2$. But $|X_v| \leq t$, as $A_v \cong L_2(r)$, $r \leq t$, a contradiction. □

Lemma 4.28. *Assume Hypothesis* 4.23. *If* $L \in \bar{\mathcal{L}}^*_{A_z}$, *then* $L \not\cong Sp_{2n}(2)$.

Proof. Assume false. Let $A_z = K_1, \ldots, K_r = L$ be a chain belonging to $A_z \to L$. By Hypothesis 4.23 we have $A_z \neq L$. Hence by Lemma 4.6(b) there is some involution t in $\mathrm{Aut}(L)$ such that $C_L(t)$ has a component K_{r-1}, which contradicts Lemma 2.25. □

We fix the following notation: Let $L \in \bar{\mathcal{L}}^*_{A_z}$ and U be a Sylow 2–subgroup of $C_G(L)$. Furthermore let T be a Sylow 2–subgroup of G such that $T \cap N_G(L)$ is a Sylow 2–subgroup of $N_G(L)$ containing U.

By Lemma 4.25 and Lemma 4.26 we may assume that $L \in \mathrm{Chev}(2)$ and by Proposition 4.5 L is standard. Further by Lemma 2.25 L does not possess an outer automorphism centralizing a Sylow 2–subgroup of L. By Hypothesis 4.23 $Z(T) \cap U = 1$. As $|\Omega_1(Z(T))| \geq 4$ by Lemma 4.24, we get that

$$|\Omega_1(Z(T \cap L))| \geq 4.$$

This gives that either L is defined over $\mathrm{GF}(q)$, $q \geq 4$, or $L \cong F_4(2)$ by Lemma 4.28. This we now collect in the following lemma

Lemma 4.29. *Assume Hypothesis* 4.23*. Then L is defined over* $\mathrm{GF}(q)$*, $q > 2$.*

Proof. Let $L \cong F_4(2)$ and $A_z = K_1, \ldots, K_r = L$ be a chain belonging to $A_z \to L$. By Lemma 4.6(b) and Hypothesis 4.23 we have that K_{r-1} is a component in the centralizer of some involution t of $\mathrm{Aut}(L)$. By Lemma 2.25 we get $K_{r-1} = {}^2F_4(2)'$. We have $t \in T$. As $Z(T)U/U$ is a fours group, we get that this group is centralized by t. But then $Z(T \cap K_{r-1})$ must contain a fours group, a contradiction. □

Lemma 4.30. *Assume Hypothesis* 4.23 *with $L \in \bar{\mathcal{L}}^*_{A_z}$.*

(a) Assume $L \cong Sp_{2n}(q)$, or $F_4(q)$, $q = 2^n$, $n \geq 2$. Then $A_z \cong Sp_{2n}(r)$, $F_4(r)$, ${}^2F_4(r)'$ or $Sz(r)$, $r = 2^t$, $r \leq q$, where in the first two cases even $r^2 \leq q$. Finally $A_z \cong Sp_{2n}(r)$ just occurs for $L \cong Sp_{2n}(q)$ and $A_z \cong F_4(r)$ or ${}^2F_4(r)'$ just occure for $L \cong F_4(q)$.

(b) $L \not\cong Sz(q)$ or ${}^2F_4(q)'$.

(c) If $L \cong L_3(q)$ or $U_3(q)$, $q = 2^n$, then $A_z \cong L_2(r)$, $L_3(r)$ or $U_3(r)$, $r = 2^t$, where $t \leq n$ in the first case and $2t \leq n$ in the last two cases.

Proof. (a) Let first K be a central extension of one of the groups $Sp_{2n}(r)$, $F_4(r)$, ${}^2F_4(r)'$ or $Sz(r)$ and $K_1 \in \mathcal{C}_2$ with $K_1 \sqsubseteq K$. Then there is some involution t normalizing K such that $C_K(t)$ has a component K_1. Now by Lemma 2.25 (3) $K_1 \cong Sp_{2n}(s)$, $F_4(s)$, ${}^2F_4(r)$, or $Sz(r)$ in case of $K \cong Sp_4(r)$, or a central extension of such a group, where $s^2 \leq r$. Furthermore $F_4(s)$ and ${}^2F_4(r)$ just occure for $F_4(r)$. Hence we see that if $A_z \to L$, then A_z is a central extension of one of the groups of the assertion. As $Z(A_z) = 1$ by Hypothesis 4.23, we have the assertion.

Suppose finally that $L \cong Sz(q)$. Then $L = A_z$ as $\mathrm{Aut}(L)$ contains no involution with nonsolvable centralizer by Lemma 2.25.

Similarly one gets (c) by quoting Lemma 2.25(5) again. □

For the remainder of this section we fix the following notation. We choose $v \sim z$, $v \in Z(T)$. Recall that T is a Sylow 2-subgroup of G containing a Sylow 2-subgroup of $N_G(L)$, which contains U.

Lemma 4.31. *If $L/Z(L) \cong L_3(4)$ or $G_2(4)$, then $Z(L) = 1$, $L \cong L_3(4)$, $A_v \cong A_5$ and A_v is normalized by $T \cap N_G(L)$.*

Proof. Assume false. Let $A_z = K_1, \ldots, K_r = L$ be the chain belonging to $A_z \to L$. Then there is some involution $t \in T \cap N_G(L)$ such that $[L, t] = L$ and t centralizes $Z(T \cap L/Z(L))$, which is a fours group by Lemma 2.41 and Lemma 2.52. As by Lemma 2.52 $G_2(4)$ just has field automorphisms, which of course do not centralize $Z(T \cap L/Z(L))$, we see that $L/Z(L) \not\cong G_2(4)$. Hence $L/Z(L) \cong L_3(4)$ and in particular $C_{L/Z(L)}(t) \cong A_5$ by [GoLyS5, Lemma 10.2.1]. This shows $K_{r-1} \cong A_5$. Now we get $A_z = K_{r-1}$. We have that $v \in UL$. Further we have that t induces a graph automorphism and $T \cap N_G(L) = (T \cap UL)\langle t \rangle$. Let A_v be the component corresponding to A_z in $C_G(v)$. Then $A_v \cong A_5$.

Let $T_1 \leq T$ with $|T_1 : T \cap N_G(L)| = 2$. Then $U \cap U^g = 1$ for $g \in T_1 \setminus N_T(L)$. As $|T \cap N_G(L) : C_{T \cap N_G(L)}(u)| \leq 2$ for all $u \in U^\sharp$, we get that the same is true for all $u^g \in U^g$. As by Lemma 2.52(b) $|T \cap N_G(L) : C_{T \cap N_G(L)}(x)| \geq 2^4$ for any $x \in T$, which induces a graph automorphism on L, we see that $U^g \leq UL$. But then $U^g U/U$ is a subgroup of the root group of $T \cap L/Z(L)$, which is of order 4. Hence U is elementary abelian of order at most 4.

Let $1 \neq u \in U$. As $O(C_G(u)) = 1$, we get that $C_G(u) \cap C_G(L) = U$. As by Lemma 2.41 $C_{C_G(u)}(v)$ is solvable, we see that $[A_v, u] = A_v$.

Now set

$$L_v = \langle A_v^{T \cap N_G(L)} \rangle.$$

Suppose $A_v = L_v$. Then $T \cap N_G(L)$ induces a group of automorphisms isomorphic to a subgroup of D_8 on A_v. Suppose $Z(L) \neq 1$. We have that $Z(L) \leq U$. By Lemma 2.42(b) we then get that $Z(L) \cap C_T(A_v) \neq 1$, a contradiction. Hence $Z(L) = 1$. So we may assume that $A_v \neq L_v$. We have that $C_{A_v}(u)$ either is a fours group or isomorphic to Σ_3. As $A_v \cong A_5$ there is no direct product of groups isomorphic to Σ_3, which is normalized by $T \cap L$, we see that $C_{A_v}(u)$ is a fours group. As $L_3(4)$ by Lemma 2.41 does not contain elementary abelian subgroups of order greater than 16, there is no elementary abelian subgroup of order 2^8 in $C_G(u)$. Hence we have that L_v is a direct product of two copies of A_v.

Now $\langle v, C_{L_v}(u)\rangle$ is an elementary abelian group of order 32, which is normalized by $T \cap N_G(L)$, contradicting Lemma 2.42. $\square$

Lemma 4.32. *We have $Z(L) = 1$.*

Proof. By Lemma 4.29 we have that L is defined over $\mathrm{GF}(q)$, $q > 2$. If $Z(L) \neq 1$, we have with Lemma 2.63 that $L \cong L_3(4)$, $G_2(4)$ or $Sz(8)$. By Lemma 4.31 $L/Z(L) \not\cong G_2(4)$ and $Z(L) = 1$ for $L/Z(L) \cong L_3(4)$. As centralizers of involutions in $\mathrm{Aut}(Sz(8))$ are solvable (see Lemma 2.25(3)), we have that $L/Z(L) \not\cong Sz(8)$. $\square$

By Lemma 4.32 we now have that $LC_G(L) = L \times C_G(L)$. Let U and v be as before. Let R be a long root subgroup in L, if $L \not\cong Sp_{2n}(q)$. Let R be a short root subgroup in L if $L \cong Sp_{2n}(q)$. Let $X_R = C_L(R)$, $Q_R = O_2(X_R)$ and choose notation such that $[v, Q_R] = 1$. The structure of X_R and Q_R is given in Lemma 2.28 and will be used freely in the sequel.

Lemma 4.33. *Assume Hypothesis 4.23. If $v \in UR$, then $[A_v, u] = A_v$ for all $u \in U^\sharp$.*

Proof. As $v \in UR$ we have that $[v, X_R] = 1$. Let $u \in U^\sharp$ and assume that $C_{A_v^u A_v}(u) \cong A_v$. Then $\langle C_{A_v^u A_v}(u)^{X_R}\rangle$ is a product of quasisimple groups isomorphic to A_v and normalized by X_R. So it is contained in $C_G(L)$. Hence we may assume that $U \cap C_{A_v^u A_v}(u) \neq 1$. We have $E(C_G(v)) = A_v A_v^u C_{E(C_G(v))}(U \cap C_{A_v^u A_v}(u))$. As seen before all components of $E(C_G(v))$ which are in $C_{E(C_G(v))}(U \cap C_{A_v^u A_v}(u))$ are in fact in $C_G(L)$. So we have that $C_{A_v A_v^u}(u) C_{E(C_G(v))}(U \cap C_{A_v^u A_v}(u)) \leq C_G(L)$. As $T \leq C_G(v)$ we see that $Z(T) \cap C_{A_v A_v^u}(u) C_{E(C_G(v))}(U \cap C_{A_v^u A_v}(u)) \neq 1$. But then $C_G(L) \cap \Omega_1(Z(T)) \neq 1$, a contradiction to Hypothesis 4.23. Hence we have that $[A_v, u] = A_v$ for all $u \in U^\sharp$. $\square$

Lemma 4.34. *Assume Hypothesis 4.23. If $v \in UR$, then $[R, A_v] \neq 1$.*

Proof. Suppose false. As $v \in UR$ we have that $[v, X_R] = 1$. Let $u \in U^\sharp$. By Lemma 4.33 we have

$$[A_v, u] = A_v \text{ for all } u \in U^\sharp.$$

Set

$$L_v = \langle A_v^{X_R}\rangle.$$

As $[R, A_v] = 1$ by assumption, we have $[R, L_v] = 1$. As $Z(E(C_G(v))) = 1$, we have that $R \cap L_v = 1$. As $C_{L_v}(u)$ is X_R–invariant and does not contain R, we have with Lemma 2.39 that $C_{L_v}(u) \cap X_R$ is contained in $C_{L_v}(u) \cap R = 1$. So we get that $[X_R, C_{L_v}(u)] \leq X_R \cap C_{L_v}(u) = 1$. This shows with Lemma 2.37 $C_{L_v}(u) \leq C_G(L)R$ and further $L_v = A_v$.

By Lemma 4.29 we have that $|R| > 2$. Hence there is some $\rho \in L$, $o(\rho) = |R| - 1$, such that ρ acts transitively on $R^\sharp$. Then ρ acts on $\langle v, R\rangle$. Set

$$L_\rho = \langle A_v^\rho \rangle.$$

Then L_ρ is a direct product of conjugates of A_v, as $A_v \leq E(C_G(\langle v, R\rangle))$. Further as X_R normalizes A_v and ρ normalizes X_R, we see that $C_{L_\rho}(u)$ is X_R–invariant. Hence again by Lemma 4.33 we get that $C_{L_\rho}(u) \leq C_G(L)R$. As $[U, \rho] = 1$ we have that ρ acts on $C_{L_\rho}(u)$. As $[\rho, C_G(L)R] = R$, $C_{C_G(L)R}(\rho) = C_G(L)$ and $R \cap L_\rho = 1$, we get $C_{L_\rho}(u) \leq C_G(L)$. Hence also $C_{A_v}(u) \leq C_G(L)$. Thus there is $1 \neq \tilde{u} \in U \cap A_v$ and so $E(C_G(v)) = A_v C_{E(C_G(v))}(\tilde{u})$. Now $C_{E(C_G(v))}(\tilde{u})$ is normalized by X_R and so again $C_{E(C_G(v))}(\tilde{u}) \leq C_G(L)$. But then $U \cap Z(T) \neq 1$, a contradiction. □

Lemma 4.35. *Assume Hypothesis* 4.23. *We have* $v \notin UR$.

Proof. Assume $v \in UR$. By Lemma 4.34 we have that $[R, A_v] \neq 1$. By Lemma 4.27 we have that $L \not\cong L_2(q)$. Set

$$L_v = \langle A_v^{X_R} \rangle$$

and

$$Y_R = C_{X_R}(L_v).$$

Then Y_R is normal in X_R. Now by Lemma 2.39 we get that

$$(*) \qquad Y_R < R.$$

By Lemma 4.33 we have that $A_v = [A_v, u]$ for all $u \in U^\sharp$. Suppose $L_v \neq A_v$. We have that $C_{L_v}(u) = \langle C_{A_v}(u)^{X_R}\rangle$. Suppose furthermore that $C_{L_v}(u) \cap X_R \not\leq O_2(X_R)$. Then by Lemma 2.38 $F^*((C_{L_v}(u) \cap X_R)O_2(X_R)/O_2(X_R))$ is normal in $X_R/O_2(X_R)$ and so a product of quasisimple groups and at most one cyclic group and each is normal in $X_R/O_2(X_R)$. But as $A_v \neq L_v$, we have that $F^*((C_{L_v}(u) \cap X_R)O_2(X_R)/O_2(X_R))$ is a product of at least two groups on which X_R acts transitively, a contradiction. So we have that $C_{L_v}(u) \cap X_R \leq O_2(X_R) = Q_R$.

Suppose that $C_{A_v}(u) \cap X_R \not\leq Z(Q_R)$. By Lemma 2.43 either $Q_R \leq L_v$, or $L \cong L_n(q)$ and $C_{L_v}(u) \cap X_R$ is elementary abelian of order q^{n-1}, or $L \cong L_3(q)$ and $\Omega_1(C_{A_v}(u)) \leq R$ by Lemma 2.43. (Recall that $L \not\cong L_3(4)$ by Lemma 4.31). In the latter case $1 \neq R \cap A_v$ is centralized by X_R, contradicting $L_v \neq A_v$. Assume now $Q_R \not\leq L_v$. Let $x \in X_R$ with $A_v^x \neq A_v$. Then in $C_{L_v}(u) \cap X_R$ we have at least two X_R–orbits, one with representative in $C_{A_v}(u)$ and one with representative in $C_{A_v A_v^x}(u) \setminus A_v$. On

$(C_{L_v}(u) \cap X_R)^\sharp$ we see that X_R has exactly $q-1$ orbits of length 1, the elements in $R^\sharp$, and one orbit of length $q^{n-1}-q$, the elements which are not in R. This implies that one of the orbits of length one must be in $A_vA_v^x$. This then shows $L_v = A_vA_v^x$. In particular $|X_R : N_{X_R}(A_v)| = 2$. With Lemma 2.44 we now get a contradiction.

So we have $Q_R \leq L_v$. As L_v normalizes A_v, we get that $[C_{A_v}(u) \cap X_R, Q_R] = R \leq A_v$, as $C_{A_v}(u) \cap Q_R \not\leq Z(Q_R)$. But then $A_v^x \cap A_v \geq R$ for all $x \in X_R$ and so $A_v = L_v$, a contradiction.

So we have that $C_{A_v}(u) \cap X_R \leq Z(Q_R)$ and so $C_{L_v}(u) \cap X_R \leq Z(Q_R)$. As $L_v \neq A_v$ we have that $C_{A_v}(u) \cap X_R \not\leq R$. In particular $Z(Q_R) > R$, i.e. $L \cong Sp_{2n}(q)$ or $F_4(q)$. Now $A_v \cong Sp_{2n}(r)$, $Sz(r)$, $F_4(r)$ or ${}^2F_4(r)'$ by Lemma 4.30. Assume $t \in C_{L_v}(u) \cap T$, $t \notin LC_G(L)$. Then $[t, X_R] \leq X_R \cap C_{L_v}(u) \leq Z(Q_R)$. Application of Lemma 2.45 yields that $L \cong Sp_4(q)$. Now Lemma 2.16 shows that t has to induce a field automorphism on L. But then it also has to induce a field automorphism on X_R/Q_R and so on $L_2(q)$, which implies $[t, X_R] \not\leq Z(Q_R)$. So we have that $UZ(Q_R)$ contains a Sylow 2–subgroup of $C_{L_v}(u)$. But by Lemma 2.25 none of the groups A_v has an automorphism whose centralizer has abelian Sylow 2–subgroups.

We have shown that $L_v = A_v$. Now X_R/Y_R acts faithfully on A_v. By Lemma 4.31 we have that $A_v \cong A_5$ in case of $L \cong L_3(4)$. But then X_R/Y_R cannot act faithfully on A_v. So we have that $L \not\cong L_3(4)$ or $Sz(q)$. We also know by $(*)$ that Y_R is a proper subgroup of R and $O^{2'}(X_R)$ contains a Sylow 2–subgroup of L. If $L \not\cong U_3(q)$ or $L_3(q)$, i.e. X_R is nonsolvable, then A_v contains a subgroup R_v which is isomorphic to $O^{2'}(X_R/Y_R)$. Furthermore a central extension of A_v is isomorphic to a subgroup of L, as $L \in \bar{\mathcal{L}}^*_{A_z}$. So a central extension of R_v is a subgroup of L and so contained in a parabolic, which then has to be isomorphic to X_R. This shows that $X_R \cong R_v$. As $L \in \bar{\mathcal{L}}^*_{A_z}$, we now see that L has an involutory automorphism t whose centralizer in L has a component and t centralizes a Sylow 2–subgroup of L, contradicting Lemma 2.25.

So we are left with $L \cong L_3(q)$ or $U_3(q)$. Now by Lemma 4.30 we see that A_v might be $L_2(r)$, $L_3(r)$ or $U_3(r)$, where $r \leq \sqrt{q}$ in the last two cases and $r \leq q$ in the first one. But no such group has an automorphism group of order $|X_R/Y_R| \geq 2q^2$ by Lemma 2.53. $\square$

Lemma 4.36. *Assume Hypothesis 4.23. Then $L \cong Sp_{2n}(q)$ or $F_4(q)$ and $A_v \cong Sp_{2n}(r)$, $Sz(r)$, $F_4(r)$ or ${}^2F_4(r)'$.*

Proof. By Lemma 4.35 we have $v \notin UR$. Hence $Z(T) \not\leq UR$. Then $Z(T \cap L) > R$. Now by Lemma 2.32 we have that $L \cong Sp_{2n}(q)$ or $F_4(q)$. The assertion follows with Lemma 4.30. $\square$

Lemma 4.37. *Assume Hypothesis* 4.23. *Then* $L \cong Sp_4(q)$.

Proof. Assume $L \ncong Sp_4(q)$. Set $T_R = C_L(\langle R, v\rangle)^\infty$. As $L \ncong Sp_4(q)$, and $L \cong Sp_{2n}(q)$ or $F_4(q)$, we have by Lemma 2.29 or Lemma 2.28 that $T_R \neq 1$. We collect some properties of T_R which can be read of from Lemma 2.29 or Lemma 2.28.

(1) $T_R/Q_R \cong Sp_{2n-4}(q)$ in case of $L \cong Sp_{2n}(q)$.
(2) If $L \cong F_4(q)$ we have that $T_R/O_2(T_R) \cong Sp_4(q)$. Furthermore we have
   (i) $O_2(T_R) = Q_R Q_{R_1}$, where R_1 is a root group such that $Z(T \cap L) = RR_1$.
   (ii) $O_2(T_R)' \leq Q_R \cap Q_{R_1}$.

Set
$$L_v = \langle A_v^{T_R} \rangle$$
and
$$Y_R = C_{T_R}(L_v).$$
Assume first $[R, A_v] = 1$. Then $R \cap L_v = 1$ by Hypothesis 4.23. Furthermore for $u \in U^\sharp$ we have that $X_R \cap C_{L_v}(u) \leq Z(Q_R)$ by Lemma 2.46 and so $C_{A_v}(u)$ has a Sylow 2–subgroup in $Z(Q_R)U$. Hence $C_{A_v}(u)$ has abelian Sylow 2–subgroups, a contradiction to Lemma 2.25.

So we have $[R, A_v] \neq 1$. Still $C_{L_v}(u)$ is normalized by T_R and so it cannot have a subgroup isomorphic to A_v as this subgroup has to be in $C_G(L)$ but A_v does not have abelian Sylow 2–subgroups. So we have that $A_v = [A_v, u]$ for all $u \in U^\sharp$.

We have $Y_R \trianglelefteq T_R$. As $R \not\leq Y_R$, we see with Lemma 2.46 that $Y_R \leq Z(Q_R)$. So we obtain that:

$$[R, A_v] \neq 1 \text{ and } Y_R \leq Z(Q_R). \tag{1}$$

Suppose first $L_v \neq A_v$. As $T_R/O_2(T_R) \cong Sp_{2n-4}(q)$ or $Sp_4(q)$ is simple, recall that by Lemma 4.29 $q > 2$, we get $C_{L_v}(u) \cap T_R \leq O_2(T_R)$. If $R \cap A_v \neq 1$, then as $[R \cap A_v, T_R] = 1$, we get a contradiction to $A_v \neq L_v$. So

$$R \cap A_v = 1. \tag{2}$$

Firstly we consider $L \cong Sp_{2n}(q)$. Assume $A_v \cap Q_R \not\leq Z(Q_R)$. As $[L_v \cap Q_R, A_v \cap Q_R] \leq Q_R' = R$, we see with (2) that $A_v \cap Q_R$ is centralized by $L_v \cap Q_R$ and so $L_v \cap Q_R$ is abelian. As $L_v \cap Q_R \trianglelefteq T_R$ we have that $Q_R \not\leq N_G(A_v)$. Let $t \in Q_R$ with $A_v^t \neq A_v$. As $1 \neq [A_v \cap Q_R, t] \leq Q_R' = R$, we have $R \cap A_v A_v^t \neq 1$, $R \cap A_v A_v^t$ is centralized by T_R so $A_v A_v^t = L_v$. In particular $|Q_R : N_{Q_R}(A_v)| = 2$. But then by (2) we have that $N_{Q_R}(A_v)$ centralizes $A_v \cap Q_R$, which contradicts $q > 2$. So we have that $A_v \cap Q_R \leq Z(Q_R)$ and thus $Q_R \cap L_v \leq Z(Q_R)$. But now also $O_2(T_R) \cap L_v \leq Z(Q_R)$. So $C_{A_v}(u)$ has a Sylow 2–subgroup contained in $UZ(Q_R)$, which contradicts Lemma 2.25

Hence we have that $L \cong F_4(q)$. As $T_R \cap L_v$ is contained in a parabolic subgroup of L_v by Lemma 4.36 and the Borel-Tits-Theorem 2.15 we see with Lemma 4.30 that $T_R \cap L_v \leq O_2(T_R)$. As $A_v \neq L_v$ and $[Z(T \cap L), T_R] = 1$, we obtain

$$Z(T \cap L) \cap A_v = 1. \tag{3}$$

If $Q_R \leq L_v$, then as $Z(T \cap L) \cap A_v = 1$ we conclude $Q_R \cap A_v \leq Z(Q_R) \cap A_v$. Let $t \in C_{A_v}(u)$. Then $[t, Q_R] \leq Z(Q_R)$ and as $Q_R \cap A_v \leq Z(Q_R)$ we get $t \in Q_R U$ and then $t \in Z(Q_R)U$, which gives that $C_{A_v}(u)$ has an abelian Sylow 2–subgroup, a contradiction to Lemma 2.25.

So we have that

$$Q_R \not\leq L_v \text{ and then } O_2(T_R) \not\leq L_v. \tag{4}$$

We further have that $C_{A_v}(u)$ does not have an elementary abelian Sylow 2–subgroup. Hence $O_2(T_R) \cap A_v$ is not abelian. As $Z(T \cap L) \cap A_v = 1$ by (3), we have that $Q_R \cap A_v$ is elementary abelian.

Assume first that $Q_R \cap A_v = 1$. As $O_2(T_R)' \leq Q_R$, we get that $O_2(T_R) \cap A_v$ is abelian, a contradiction. So we have that

$$Q_R \cap A_v \neq 1. \tag{5}$$

Suppose that $O_2(T_R)$ normalizes A_v. Then $Z(O_2(T_R)) \cap A_v \neq 1$. But $Z(O_2(T_R)) = Z(T \cap L)$, a contradiction. As $O_2(T_R) = Q_R Q_{R_1}$, where R_1 is a root group different from R in $Z(T \cap L)$, we may assume that Q_R does not normalize A_v. In this case we see that $Z(Q_R) \cap A_v = 1$, as Q_R centralizes this group. As $A_v \cap Q_R \neq 1$ by (5), and $[Q_R \cap A_v, N_{Q_R}(A_v)] = 1$ by (3), we see that $|Q_R : N_{Q_R}(A_v)| \geq q$. As $[Q_R, Q_R \cap A_v] \leq R$, we have that $|\langle (Q_R \cap A_v)^{Q_R} \rangle| \leq q|Q_R \cap A_v|$. On the other hand we have at least q conjugates of A_v under Q_R, so

$$|\langle (Q_R \cap A_v)^{Q_R} \rangle| \geq |Q_R \cap A_v|^q.$$

This shows $|Q_R \cap A_v|^{q-1} \leq q$. As $q > 2$ and $|Q_R \cap A_v| \neq 1$, this is a contradiction.

So we have shown that

$$A_v = L_v. \tag{6}$$

By (1) there is some subgroup W of $Z(Q_R)$ such that T_R/W acts faithfully on A_v. As T_R/W possess no solvable factor groups and the outer automorphism group of A_v is solvable we get that T_R/W acts as a group of inner automorphism. As $|T_R/Z(Q_R)|_2 = q^{n^2-4}$ in case of $L \cong Sp_{2n}(q)$ and $|T_R/Z(Q_R)|_2 = q^{17}$ in case of $L \cong F_4(q)$, we get $|A_v|_2 \geq q^{n^2-4}$ in case of $Sp_{2n}(q)$ and $|A_v|_2 \geq q^{17}$ in case of $F_4(q)$. On the other hand by Lemma 4.30 we have that $A_v \cong Sp_{2n}(r)$, $F_4(r)$, ${}^2F_4(r)'$ or $Sz(r)$. Hence in this ordering we get $|A_v|_2 = r^{n^2}$, r^{24}, r^{12}, or 2^{11}, or r^2. Furthermore $|A_v|_2 \leq |L|_2$ by Lemma 4.6. If $A_v \cong Sz(r)$, then $q = r$ which violates the inequalities above. If $L \cong Sp_{2n}(q)$, we have $|A_v| \leq q^{n^2}$. On the other hand for $A_v \cong Sp_{2n}(r)$ we get $r^2 \leq q$, so $|A_v|_2 = r^{n^2} \leq q^{n^2/2}$. Then $n^2 - 4 \leq n^2/2$, which gives the contradiction $n \leq 2$. As $F_4(r)$ and ${}^2F_4(r)'$ do not show up for $L \cong Sp_{2n}(q)$, we have to deal with $L \cong F_4(q)$ in which case $|A_v|_2 \geq q^{17}$. Suppose $A_v \cong F_4(r)$, then again $r^2 \leq q$ and so $|A_v|_2 = r^{24} \leq q^{12}$, a contradiction. If $|A_v| \cong {}^2F_4(r)$, then $r \leq q$ and $|A_v| \leq r^{12} \leq q^{12}$, a contradiction. □

Lemma 4.38. *Hypothesis* 4.23 *does not hold.*

Proof. Suppose Hypothesis 4.23 holds. Then by Lemma 4.37 we have that $L \cong Sp_4(q)$, $q > 2$. Further $A_v \cong Sp_4(r)$ or $Sz(r)$ by Lemma 4.30. By Lemma 4.35 $v \notin UR$. Now set

$$L_v = \langle A_v^{T \cap L} \rangle.$$

Then we have that $C_{L_v}(u)$ is normalized by $T \cap L$ for $u \in U^\sharp$. As $T \cap L$ does not centralizes any perfect subgroup of $N_G(L)$ which is centralized by $\langle u, v \rangle$, and A_v has nonabelian Sylow 2–subgroups, we get that $[A_v, u] = A_v$ for all $u \in U^\sharp$. Further by Lemma 2.25 $C_{A_v}(u)$ has a Sylow 2–subgroup, which is not elementary abelian.

Suppose first that $A_v \neq L_v$. Then as before

$$Z(L \cap T)U \cap A_v = 1. \tag{1}$$

Further we may assume that $T \cap N_G(L)$ contains a Sylow 2–subgroup of $C_{A_v}(u)$. Assume $T \cap L \cap A_v = 1$. Then $[N_{T\cap L}(A_v), T \cap C_{A_v}(u)] = 1$. Choose $1 \neq x \in T \cap C_{A_v}(u)$. As for any outer automorphism $x \in T$ of L, we have by Lemma 2.49 that $|T \cap L : C_{T\cap L}(x)| \geq q^2$, we get that $|T \cap L : N_{T\cap L}(A_v)| \geq q^2$. Hence now there are at least q^2 conjugates of

A_v under the action of $T \cap L$. So we get

$$|N_G(L) \cap T \cap A_v|^{q^2} \leq |N_G(L) \cap T \cap A_v| q^2,$$

which is impossible.

So we have that $T \cap L \cap A_v \neq 1$. As $(T \cap L)' = Z(T \cap L)$ by Lemma 2.48, we get with (1) that $N_{T \cap L}(A_v)$ is elementary abelian, and so $|T \cap L : N_{T \cap L}(A_v)| \geq q$ by Lemma 2.48. Now we have at least q conjugates of A_v under the action of $T \cap L$, which yields

$$|T \cap L \cap A_v|^q \leq |T \cap A_v \cap L| q^2.$$

This gives $q = 4$ and $|T \cap L \cap A_v| = 2$. As $UZ(T \cap L) \cap A_v = 1$ by (1), we see that $|N_G(L) \cap T \cap A_v| \leq 4$, which gives that $C_{A_v}(u)$ has abelian Sylow 2–subgroups, a contradiction. This shows

$$A_v = L_v.$$

Assume $[Z(T \cap L), A_v] = 1$ As $T \cap L$ normalizes A_v we get $[T \cap L \cap A_v, T \cap L] \leq Z(T \cap L) \cap A_v \leq Z(A_v) = 1$ by Hypothesis 4.23. Then we have hat $T \cap L \cap A_v \leq Z(T \cap L) \cap A_v \leq Z(A_v) = 1$. But then $C_{A_v}(u) \leq Z(T \cap L) C_G(L)$ and so it has abelian Sylow 2–subgroups, a contradiction. So we have

$$[Z(T \cap L), A_v] \neq 1.$$

As by Lemma 2.48 $Z(T \cap L)$ is a product of two root groups, we may assume $[R, A_v] \neq 1$. First assume $A_v \cong A_6$. Then we have that $C_{A_v}(u)$ is cyclic of order 4 or dihedral of order 8. But by Lemma 2.50 no such group can be normalized by $T \cap L$. So we have that $A_v \cong Sp_4(r)$ or $Sz(r)$ with $r > 2$. In particular $q > 4$. As $C_{T \cap L}(A_v) \trianglelefteq T \cap L$, we get with Lemma 2.50(ii) that $|T \cap L : C_{T \cap L}(A_v)| \geq q^2/2$.

Suppose first that $A_v \cong Sp_4(r)$, $r > 2$. We will show that $\bar{T} = T \cap L / C_{T \cap L}(A_v)$ satisfies the assumptions of Lemma 2.51. Assumption (i) is clear as $T \cap L$ is generated by involutions according to Lemma 2.48. Furthermore T and so $\bar{T}$ is of class two and so if it contains a Sylow 2-subgroup of A_v it is a Sylow 2-subgroup of A_v, as otherwise by Lemma 2.16 it would also contain a field automorphism of A_v and so cannot be of class two. Hence we have $|\bar{T}| \leq r^4$. As $r^2 \leq q$ we have hat $|\bar{T}| \geq r^4/2$. Hence (ii) is satisfied. Furthermore $q^2/2 \leq |\bar{T}| \leq q^2$ and so the assumptions of Lemma 2.50(ii) are satisfied, which implies that Lemma 2.51(iii) and (iv) are satisfied. Now application of Lemma 2.51 yields the contradiction $r = 2$.

So we have that $A_v \cong Sz(q)$, $q = r^2$. Then $T \cap L$ just induces inner automorphism on A_v. In particular $T \cap L/C_{T\cap L}(A_v)$ is isomorphic to a subgroup of index one or two of a Sylow 2–subgroup of A_v. But T is generated by involutions, a contradiction to $|\Omega_1(T \cap L)| = q$. □

Proposition 4.39. *There is some $1 \neq z \in \Omega_1(Z(S))$ such that $C_G(z)$ possesses a standard component L.*

Proof. We have the assertion with Proposition 4.22 in case of $1 \neq Z(E(C_G(z))$ for at least one $1 \neq z \in \Omega_1(Z(S))$ and with Lemma 4.38 otherwise. □

Proposition 4.40. *Let z, L be as in Proposition 4.39. Then $C_G(L)$ has cyclic Sylow 2–subgroups and $C_{C_G(L)}(u)$ is a 2–group for any involution $u \in C_G(L)$.*

Proof. By Proposition 3.5 we have $m_2(C_G(L)) = 1$. By Lemma 3.2 we have that $C_G(L)$ has a cyclic Sylow 2–subgroup. As $O(C_G(u)) = 1$ for any involution $u \in C_G(L)$, we get that $C_{C_G(L)}(u)$ is a 2-group. □

Now Proposition 4.39 and Proposition 4.40 prove Proposition 4.1. From this also Theorem 1.4 follows.

References

[Asch1] M. Aschbacher, On finite groups of component type, Ill. J. Math. 19 (1975), 87 - 115.

[Asch2] M. Aschbacher, A characterization of Chevalley groups over fields of odd order, I, II, Ann. of Math. 106 (1977), 353 - 468, Correction, Ann. of Math. 111 (1980), 411 - 414.

[Asch3] M. Aschbacher, Standard components of alternating type centralized by a 4–group, J. Algebra 319 (2008), 595 - 616.

[Asch4] M. Aschbacher, 3–Transposition Groups, Cambridge Universtity Press, 1997.

[Asch5] M. Aschbacher, Sporadic groups, Cambridge University Press, 1994.

[Asch6] M. Aschbacher, The uniqueness case for finite groups I, II, Ann. Math. 117 (1983), 383 -454, 455-551.

[AschSe1] M. Aschbacher, G. Seitz, On finite groups with a standard component of known type, Osaka J. Math. 13 (1976), 439 - 482.

[AschSe2] M. Aschbacher, G. Seitz, On finite groups with a standard component of known type II, Osaka J. Math. 18 (1981), 703 - 723.

[AschSe3] M. Aschbacher, G. Seitz, Involutions in Chevalley groups over fields of even order, Nagoya Math. J. 63 (1976), 1 - 91; Correction, Nagoya Math. J. 72 (1978), 135 - 136.

[AS] M. Aschbacher, S. Smith, The classification of quasithin groups I, II, Mathematical Surveys and Monographs 111, 112, AMS 2004.

[AzBaSei] H. Azad, M. Barry, G. Seitz, On the structure of parabolic subgroups, Comm. Algebra 18 (1990), 551-562.

[CCNPW] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, Atlas of finite groups, Clarendon Press, Oxford 1985.

[CurKaSei] C. Curtis, W. Kantor, G. Seitz, The 2-Transitive Permutation Representations of the Finite Chevalley Groups, Trans. of the AMS 218 (1976), 1-59.

[DaSo] S. Davis, R. Solomon, Some sporadic characterizations, Comm. Algebra 9 (1981), 1725 - 1742.

[DeSte] A. Delgado, D. Goldschmidt, B. Stellmacher, "Groups and Graphs: New Results and Methods, DMV Seminar 6 (1985), Birkhäuser.

[FT] W. Feit, J. Thompson, Solvability of groups of odd order, Pac. J. Math. 13 (1963), 775 - 1029.

[Glau] G. Glaubermann, Central elements in core-free groups, J. Algebra 4 (1966), 403 - 421.

[GoLy] D. Gorenstein, R. Lyons, Finite groups of characteristic 2 type, Mem. of AMS 276, 1983.

[GoLyS1] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Amer. Math. Soc. Surveys and Monographs 40 (1) (1994).

[GoLyS2] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Amer. Math. Soc. Surveys and Monographs 40 (2) (1996).

[GoLyS3] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Amer. Math. Soc. Surveys and Monographs 40 (3) (1998).

[GoLyS4] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Amer. Math. Soc. Surveys and Monographs 40 (4) (1999).

[GoLyS5] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Amer. Math. Soc. Surveys and Monographs 40 (6) (2000).

[GM] R.M. Guralnick, G. Malle, Classification of 2F-Modules, I, J. Algebra 257, 2002, 348 - 372.

[GM1] R.M. Guralnick, G. Malle, Classification of 2F-Modules, II, J. Algebra 257, 2002, 348 - 372.

[GLM] R. M. Guralnick, R. Lawther, G. Malle, The 2F–modules for nearly simple groups, J. Algebra 207 (2007), 643 - 676. (2008), 621 - 629.

[Hu] B. Huppert, Endliche Gruppen I, Springer 1967.

[Ja] Z. Janko, A new finite simple group with abelian 2–Sylow subgroup and its characterization, J. Alg. 3 (1966), 147 - 186.

[KuSte] H. Kurzweil, B. Stellmacher, The theory of finite groups, An introduction, Springer 2004.

[MeStStr] U. Meierfrankenfeld, B. Stellmacher, G. Stroth, The structure theorem for finite groups with a large p-subgroup, Memoirs of AMS, to appear.

[Shi] K. Shinoda, Conjugacy classes of Chevalley groups of type (F_4), J. of Faculty of Science Univ. Tokyo 21, (1974), 133-159.

[Sm] S. Smith, Irreducible modules and parabolic subgroups, J. Algebra 75, 286 - 289, 1982.

[We] R. Weiss, The automorphism group of ${}^2F_4(2)'$, Proc. AMS 66 (1977), 208 - 210.

KAY MAGAARD, SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDGBASTON, BIRMINGHAM B15 2TT, UNITED KINGDOM
E-mail address: k.magaard@bham.ac.uk

GERNOT STROTH, INSTITUT FÜR MATHEMATIK, UNIVERSITÄT HALLE - WITTENBERG, THEORDOR LIESER STR. 5, 06099 HALLE, GERMANY
E-mail address: gernot.stroth@mathematik.uni-halle.de